CIT 5920–Mathematical Foundations of Computer Science
Final Exam

Version: December 10, 2024

Exam Instructions

- **PLEASE DO NOT BEGIN UNTIL EXPLICITLY INSTRUCTED TO DO SO.**
- **No Leaving Early:** To ensure everybody can take the exam in a quiet environment, we ask that you remain in the room until the end of the exam. If you finish early, you can use the remaining time to review your answers.
- **Anonymity:** To ensure fairness in grading, please refrain from writing your name on the exam. We will grade papers anonymously, but rest assured that your final grade will be recorded accurately.
- **Electronic Devices:** Set your cellphones to “Do Not Disturb” mode. Calculators are not permitted during this exam.
- **Simplification:** It’s okay to leave your answers in an unsimplified form. For instance, you don’t need to simplify “2 + 2”. We’re assessing your understanding, not your mental arithmetic skills.
- **Answer Space:** Ensure you provide your answers within the designated spaces. We won’t check the back of any page.
- **Scratch Paper:** There’s a blank sheet provided at the end of the exam for any rough work. Additionally, feel free to use the back of any sheet if you need more space.

1. Discrete Mathematics Notation Pot Pourri (30 Points)
2. Counting Mathematical Objects: Integers, Relations, Functions (24 Points)
3. Calculating Conditional Probabilities (4 Points)
4. Revisiting Sum of Digits and Linearity of Expectation (6 Points)
5. Evaluating the Statements ChatGPTv4 Makes About Logic Statements (12 Points)
6. Guided Weak Induction on Sets (20 Points)
7. Weak Induction On Your Own (10 Points)
8. Graph Concepts (10 Points)

Your **PennID** (the 8 **digits** in big font on your penncard): ______________________________

A big shout out to the amazing team of TAs—Jordan, Akanksha, Armaan, Binbin, Felix, Grace, and Lang—for their stellar support throughout the term.

It was a pleasure, an honor and incredibly stimulating to be your Professor this semester. Thank you for your engagement. Thank you also for appreciating my unique way of doing things, and for providing me many opportunities to create original outputs for the class. I hope this exam feels like an appropriate assessment to celebrate your learning in this course.

GOOD LUCK!

Exercise 1 – Discrete Mathematics Notation Pot Pourri [30pts]

Translate the following statements into formal mathematical notation:

(i) "The number of ways to choose 3 items from 10 is C."

(ii) "The statement P implies the statement Q."

(iii) "The relation R on set A is reflexive."

(iv) "The logical negation of proposition P is true."

(v) "There exists an element in set X for which predicate $P(x)$ is true."

(vi) "For all elements in set X, the predicate $P(x)$ is false."

(vii) "The binary relation R on set X is transitive."

(viii) "The logical conjunction of propositions P and Q is false."

(ix) "For every vertex x in graph G, there exists an edge connecting it to vertex y."

(x) "For every integer x, there exists an integer y such that x is less than y."

Exercise 2 – Counting Mathematical Objects: Integers, Relations, Functions [24pts]

A. How many 10 digit numbers are there that have exactly four 1s?

B. Let $S = \{x, y\}$, how many relations on S are reflexive? You can provide a number, a formula, an explanation, or a combination of these.

C. How many onto functions are there from $\{a, b, c, d, e\}$ to $\{p, q\}$? Do this without having to explicitly list out all functions.

Exercise 3 – Calculating Conditional Probabilities [4pts]

A tech company has 200 employees, categorized based on their primary skill set: 120 are in software development and 80 in data analysis. Among the software developers, 40 have leadership roles, while 20 data analysts are in leadership positions. Selecting an employee randomly:

A. Calculate the conditional probability that the employee is in a leadership role, given they are a software developer.

B. Calculate the conditional probability that the employee is a software developer, given they are in a leadership role.

Exercise 4 – Revisiting Sum of Digits and Linearity of Expectation [6pts]

Recall that in the previous midterm, we had the following problem: The digits 1, 2, 3, 4, 5, and 6 are randomly arranged to form two three-digit numbers. Each digit can only be used once. For example if one of the three digit numbers is 421 then the other three digit number is some permutation of 3, 5, and 6. What is the expected value of the ***sum*** of the two numbers?

One possible solution to this problem relies on using the *linearity of expectation*:

> To solve this, we first define our random variables:
>
> - Let a, b, and c be the digits of the first three-digit number. Each of these is a random variable representing a digit.
> - Similarly, let d, e, and f be the digits of the second three-digit number.
>
> Each of these random variables (a, b, c, d, e, f) can take on the values 1, 2, 3, 4, 5, or 6, with each value being equally likely. This means each digit has a probability of $\frac{1}{6}$ of being chosen for any position.
> The three-digit numbers can be expressed in a standard form:
>
> - The first number is $100a + 10b + c$. Here, a is in the hundreds place, b in the tens place, and c in the units place.
> - The second number is $100d + 10e + f$, with d, e, and f occupying the hundreds, tens, and units places, respectively.
>
> To find the expected sum of these numbers, we use the linearity of expectation, which allows us to add the expected values of individual variables:
>
> $$E[100a + 10b + c + 100d + 10e + f] = 100E[a] + 10E[b] + E[c] + 100E[d] + 10E[e] + E[f]$$
>
> Since the digits are uniformly distributed (each having an equal chance of being any of the six digits), the expected value for each digit is the mean of the digits 1 through 6:
>
> $$E[a] = E[b] = E[c] = E[d] = E[e] = E[f] = \frac{1 + 2 + 3 + 4 + 5 + 6}{6} = 3.5$$
>
> Substituting this into our equation, we calculate the expected sum:
>
> $$2 \times (100 \times 3.5 + 10 \times 3.5 + 3.5) = 2 \times (350 + 35 + 3.5) = 2 \times 388.5 = 777$$
>
> Therefore, the expected sum of the two unique three-digit numbers formed from the digits 1 through 6 is 777.

Using this solution as a guideline, how would you determine the solution to this variant of the problem: The digits 0, 1, 2, 4, and 8 are randomly arranged to form two two-digit numbers. Each digit can only be used once. For example if one of the two digit numbers is 14 then the other two digit number is some permutation of 0, 2, and 8. What is the expected value of the ***sum*** of the two numbers?

Exercise 5 – Evaluating the Statements ChatGPTv4 Makes About Logic Statements [12pts]

In all of these questions x, y, and z come from the set $\{0, 1, 2, 3, 4, 5\}$.

For each of the following statements, we want to evaluate whether it is TRUE or FALSE.

(a) $\forall x \exists y, x \cdot y = 0$

(b) $\exists x \forall y, x + y = y + x$

(c) $\exists x \forall y, x \cdot y = y$

(d) $\forall x \exists y \exists z, x \cdot y = z^2$

(e) $\exists x \exists y \forall z, x + z = y$

(f) $\forall x \forall y, x^2 \geqslant y$

An enterprising student asks ChatGPTv4 to evaluate these statements and provide its best response. The AI responds with the following:

> (a) **False.** This statement claims that for every x, there exists a y such that $x \cdot y = 0$. This is false because if x is any non-zero number, there is no y (other than 0) that would make $x \cdot y = 0$.
>
> (b) **True.** This is true since addition is commutative; for any x, $x + y = y + x$.
>
> (c) **False.** There is no single x for which $x \cdot y = y$ for all y. This would only be true if $x = 1$, but the statement must hold for all y.
>
> (d) **True.** For any x, we can find y and z such that $x \cdot y = z^2$. For instance, if $y = x$ and $z = x^2$, the statement is true.
>
> (e) **False.** There are no x and y such that $x + z = y$ for all z.
>
> (f) **True.** Since all values are non-negative, any number squared will always be greater than or equal to any other number from the set.

Evaluate each response for accuracy, and provide a list of the mistakes you find, if any, with a brief explanation.

Exercise 6 – Guided Weak Induction on Sets [20pts]

Consider the following stament that we would like to prove is true for at least some integers:

> STATEMENT: For a set S with n elements, the number of subsets of S with an even number of elements is equal to the number of subsets with an odd number of elements.

For example, consider the set $S = \{1, 2, 3\}$; $X = \{1, 2\}$ and $Y = \{1, 3\}$ are both subsets of S with an *even* number of elements, while $Z = \{1, 2, 3\}$ is an example of a subsets of S with an *odd* number of elements.

A. Is the statement true for $n = 0$? Explain your reasoning.

B. Is the statement true for $n = 1$? Explain your reasoning.

C. To understand what happens to this property when adding an element to a set, it is useful to consider a small example. Consider a set $S = \{1, 2\}$.

   (i) List all subsets of S and determine the parity of the number of elements in each subset (that is, whether the number of elements is even or odd).

   (ii) Now, add a new element, 3, to each subset from (i), and rewrite the augmented subsets. (Note that this is different from asking for the subsets of $S \cup \{3\}$.)

(iii) How does the addition made in (ii) change the parity of each subset? What can be concluded about the parity of subsets in $P(S)$ from (i), and those in $P(S)$ augmented by $\{3\}$ from (ii)?

D. Prove the statement using weak induction.

Exercise 7 – Weak Induction On Your Own [10pts]

Prove the following statement using weak induction:

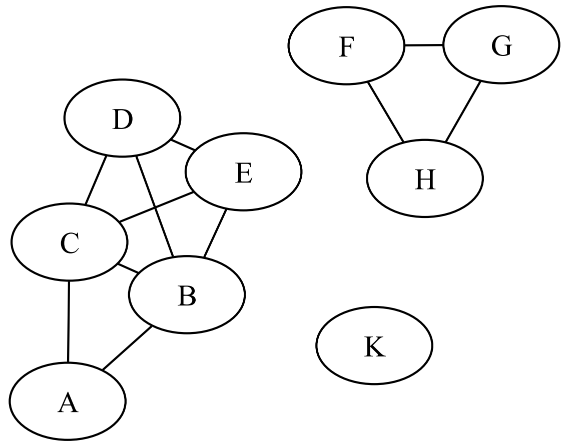
> STATEMENT: For a set S with n elements, there are $n!$ (factorial of n) ways to arrange the elements of S.

For example, for the set $\{\text{pizza}, \text{sushi}, \text{taco}\}$, which contains three elements, the arrangements are:

- pizza, sushi, taco
- pizza, taco, sushi
- sushi, pizza, taco
- sushi, taco, pizza
- taco, pizza, sushi
- taco, sushi, pizza

There are $3! = 6$ arrangements, which is consistent with our statement.

Exercise 8 – Graph Concepts **[10pts]**
Consider the following graph $G = (V, E)$:



A. Based on the image above, write the contents of the sets V and E.

B. Provide the degree sequence of this graph.

C. Does the graph contain a cycle? If so provide an example of a cycle in the graph. If not, explain why not.

D. What does it mean for a graph to be *connected*? Is the graph G connected? If so, explain why. If not, explain why not.

E. Draw a graph that is isomorphic to G but does not look exactly identical: Different edges must be crossing.

Compendium of Formulas

Set Identities

Name	Identities	
Idempotent laws	$A \cup A = A$	$A \cap A = A$
Associative laws	$(A \cup B) \cup C = A \cup (B \cup C)$	$(A \cap B) \cap C = A \cap (B \cap C)$
Commutative laws	$A \cup B = B \cup A$	$A \cap B = B \cap A$
Distributive laws	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Identity laws	$A \cup \emptyset = A$	$A \cap U = A$
Domination laws	$A \cap \emptyset = \emptyset$	$A \cup U = U$
Double complement law	$\overline{\overline{A}} = A$	
Complement laws	$A \cap \overline{A} = \emptyset$ $\overline{U} = \emptyset$	$A \cup \overline{A} = U$ $\overline{\emptyset} = U$
De Morgan's laws	$\overline{A \cup B} = \overline{A} \cap \overline{B}$	$\overline{A \cap B} = \overline{A} \cup \overline{B}$
Absorption laws	$A \cup (A \cap B) = A$	$A \cap (A \cup B) = A$

Relations

A relation R on a set X is a subset of the Cartesian Product, $X \times X$.

- A relation is *symmetric* if and only for every a that is related to b, b is related to a.
- A relation on set X is *reflexive* if and only if for every $x \in X$, (x, x) is in the relation.
- A relation R is *transitive* if and only if aRb and bRc gives us aRc.
- A relation is antisymmetric if aRb and bRa only happens when a and b are equal.

Properties of Functions

- A function $f : A \to B$ is *one-to-one* (or injective) if and only if for every $a_1, a_2 \in A$, $f(a_1) = f(a_2)$ implies $a_1 = a_2$. In other words, no two different elements in A map to the same element in B.
- A function $f : A \to B$ is *onto* (or surjective) if and only if for every $b \in B$, there exists an $a \in A$ such that $f(a) = b$. This means every element in B is the image of at least one element in A.
- A function is *bijective* if it is both one-to-one and onto. This implies a perfect ”pairing” between the elements of the sets A and B.

Counting

- The number of ways to choose k items out of n is
$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$
- The number of ways to pick and arrange k items out of n is $P(n,k) = n!/k!$.
- $|P(A)| = 2^{|A|}$.
- Number of ways to arrange n items out of which k_1 are identical of one kind and k_2 are identical of another kind is.
$$\frac{n!}{k_1!k_2!}$$

Probability Formulas

- Conditional Probability: $\mathbb{P}[A|B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]}$
- Random Variable: A random variable X is a function that assigns a real number to each outcome in the sample space of a random experiment.
- Expectation: $E[X] = \sum_{x \in X} x \cdot \mathbb{P}[X = x]$
- Linearity of Expectation: $E[X+Y] = E[X] + E[Y]$

Definitions

- A positive integer n is a perfect square, iff $\exists k \in \mathbb{Z}$ such that $n = k^2$.
- A positive integer n is composite, iff $\exists a, b \in \mathbb{Z}$ such that $n = ab$ and $1 < a < n$ and $1 < b < n$.
- A positive integer $n > 1$ is prime iff the only factors of n are 1 and n.
- For integers a and b, $a|b$ (a divides b) iff $\exists k \in \mathbb{Z}$ such that $b = ak$.
- An integer n is even iff $2|n$.
- An integer n is even iff $\exists k \in \mathbb{Z}$ such that $n = 2k$.
- An integer b is odd iff $\exists k \in \mathbb{Z}$ such that $b = 2k + 1$.
- $\forall x \in \mathbb{Z}$ x is even iff x is not odd.
- $\binom{n}{k} = \frac{n!}{k!(n-k)!}$
- The emptyset $\emptyset$ is a subset of every set.