CIT 5920 - Recap Lecture

PROBABILITY

1. Experiment — rolling two 6-sided die RED and BLUE
2. Sample Space (S) — (k, m) ordered pairs $|S| = 6 \times 6 = 36$ (uses product rule!)
3. Event (E) — "getting a sum of 7"
4. Probability (of an Event) — $\mathbb{P}[E] = \mathbb{P}[k+m=7]$

$E = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$

$\mathbb{P}[E] = \frac{|E|}{|S|} = \frac{6}{36}$

event E is a subset of S: $E \subseteq S$

Complement: $S \setminus E$ ← set difference

5. Conditional Probability — $\mathbb{P}[A \mid B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]}$

"getting a sum of 7 given that RED is 1"

$\mathbb{P}[E \mid \text{RED is } 1] = \frac{\mathbb{P}[E \cap \text{RED is } 1]}{\mathbb{P}[\text{RED is } 1]} = \frac{1/36}{1/6} = \frac{1}{36} \cdot \frac{6}{1} = \frac{1}{6}$

$E \cap \text{RED is } 1 = \{(1,6)\}$ ← this is an event too

$\mathbb{P}[E \cap \text{RED is } 1] = \frac{1}{36}$

6. Random Variable — X = sum of two die, one RED and one BLUE

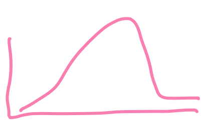
X can take value from 2 to 12 — since there are 11 possible outcomes for X, but 36 outcomes for E then it indicates X will be non-uniform

$|S| = 6 \times 6 = 36$

X	subset of S	probability
2	(1,1)	1/36
3	(1,2), (2,1)	2/36
4	(1,3), (2,2), (3,1)	3/36
5	(1,4), (2,3), (3,2), (4,1)	4/36
6	(1,5), (2,4), (3,3), (4,2), (5,1)	5/36
7	(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)	6/36 = 1/6
8	(2,6), (3,5), (4,4), (5,3), (6,2)	5/36
9	(3,6), (4,5), (5,4), (6,3)	4/36
10	(4,6), (5,5), (6,4)	3/36
11	(5,6) (6,5)	2/36
12	(6,6)	1/36

EXAMPLE OF A NON UNIFORM DISTRIBUTION

[better not to simplify so structure can be read]



$\mathbb{P}[E] = \mathbb{P}[\text{"getting a sum of 7"}] = \mathbb{P}[X=7] = 6/36$

7. Expected Value/ Expectation

$\mathbb{E}[X] = \sum_{x \in X} x \cdot \mathbb{P}[X=x]$ ← I know it's confusing

$= 2 \cdot \mathbb{P}[X=2] + 3 \cdot \mathbb{P}[X=3] + 4 \cdot \mathbb{P}[X=4] + 5 \cdot \mathbb{P}[X=5] + 6\mathbb{P}[X=7] + 7 \cdot \mathbb{P}[X=7] + 8 \cdot \mathbb{P}[X=8] + 9 \cdot \mathbb{P}[X=9] + 10 \cdot \mathbb{P}[X=10] + 11 \cdot \mathbb{P}[X=11] + 12 \cdot \mathbb{P}[X=12]$

$= 2 \cdot 1/36 + 3 \cdot 2/36 + 4 \cdot 3/36 + 5 \cdot 4/36 + 6 \cdot 5/36 + 7 \cdot 6/36 + 8 \cdot 5/36 + 9 \cdot 4/36 + 10 \cdot 3/36 + 11 \cdot 2/36 + 12 \cdot 1/36$ ← this is ACCEPTABLE FINAL ANSWER

this would have been $= 7$

8. Linearity of Expectation

For any RV X and Y: $\mathbb{E}[aX + bY] = a\mathbb{E}[X] + b\mathbb{E}[Y]$

$a=1$, $b=1$: $\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$

Let R be the random variable of throwing a RED DIE

Let B be the random variable of throwing a BLUE DIE

Then the sum of the two dice is $X = R + B$

R	subset	probability
1	⚀	1/6
2	⚁	1/6
3	⚂	1/6
4	⚃	1/6
5	⚄	1/6
6	⚅	1/6

EXAMPLE OF A UNIFORM DISTRIBUTION (fair die)

$\mathbb{P}[R=1] = 1/6$

$\mathbb{E}[R] = 1 \cdot \mathbb{P}[R=1] + 2 \cdot \mathbb{P}[R=2] + 3 \cdot \mathbb{P}[R=3] + 4 \cdot \mathbb{P}[R=4] + 5 \cdot \mathbb{P}[R=5] + 6 \cdot \mathbb{P}[R=6]$

$= 1 \cdot 1/6 + 2 \cdot 1/6 + 3 \cdot 1/6 + 4 \cdot 1/6 + 5 \cdot 1/6 + 6 \cdot 1/6$

$= 1/6\,[1+2+3+4+5+6] = \frac{1}{6} \cdot 21 = \frac{7}{2} = 3.5$

The red die and the blue die are both fair 6-sided die and have the same probability distribution.

$\mathbb{E}[B] = \mathbb{E}[R] = 7/2$

Then because of linearity of expectation

$\mathbb{E}[X] = \mathbb{E}[R+B] = \mathbb{E}[R] + \mathbb{E}[B] = 2 \cdot \frac{7}{2} = 7$

HW4. Ex3

The digits 1, 2, 3, 4 are arranged randomly to form 2-digit numbers (each digit used once). What is the expected value of the product of the two numbers. ← WORD STATEMENT

$\boxed{A|B|C} = A \cdot 100 + B \cdot 10 + C$

translating into math notation

1. Let's call the digits A, B, C, D
2. Suppose that the numbers are $\boxed{AB}$ and $\boxed{CD}$ (wlog) (A: the digit of 10's, B: the digit of units)
3. When we multiply them, we get → $A \times 10 + B \times 1$

$(10 \cdot A + B) \times (10 \cdot C + D) = 100\,A \cdot C + 10\,A \cdot D + 10 \cdot B C + B \cdot D$ (4)

6. All choices of $\boxed{AB}$ and $\boxed{CD}$ are equally likely therefore

$\mathbb{E}[A \cdot C] = \mathbb{E}[A \cdot D] = \mathbb{E}[B \cdot C] = \mathbb{E}[B \cdot D]$

7. then we just need to calculate one of them (this is similar to how we noticed the RED and BLUE dice are both just dice and have the same distribution)
8. Now let's calculate $\mathbb{E}[A \cdot C]$ (wlog)
9. The possible products for $A \cdot C$:

$1 \cdot 2,\ 1 \cdot 3,\ 1 \cdot 4,\ 2 \cdot 1,\ 2 \cdot 3,\ 2 \cdot 4,\ 3 \cdot 1,\ 3 \cdot 2,\ 3 \cdot 4,\ 4 \cdot 1,\ 4 \cdot 2,\ 4 \cdot 3$

10. All of these are equally likely, so expected is the average.

$\mathbb{E}[A \cdot C] = \frac{1 \cdot 2 + 1 \cdot 3 + 1 \cdot 4 + 2 \cdot 1 + 2 \cdot 3 + 2 \cdot 4 + 3 \cdot 1 + 3 \cdot 2 + 3 \cdot 4 + 4 \cdot 1 + 4 \cdot 2 + 4 \cdot 3}{12} = \frac{70}{12}$

11. Using linearity of expectation:

$\mathbb{E}[100\,A \cdot C + 10\,A \cdot D + 10 \cdot BC + BD]$

$= 100 \cdot \mathbb{E}[A \cdot C] + 10\,\mathbb{E}[A \cdot D] + 10\,\mathbb{E}[B \cdot C] + \mathbb{E}[B \cdot D]$

12. Since (7) then

$\mathbb{E}[\ldots] = 100 \cdot \mathbb{E}[AC] + 10\,\mathbb{E}[AC] + 10\,\mathbb{E}[AC] + \mathbb{E}[AC]$

$= (100 + 10 + 10 + 1)\,\mathbb{E}[A \cdot C] = 121 \cdot \mathbb{E}[A \cdot C]$ ✓

EXAMPLE OF PROOF BY INDUCTION FOR PROBABILITY

Coin flip: R.V. C, $S = \{\text{heads}, \text{tails}\}$

$\mathbb{P}[C = \text{heads}] = 1/2$

$\mathbb{P}[C = \text{tails}] = 1/2$

Consider a sequence of N independent coin flips. What is the probability of getting at least 1 head in the sequence? (7:25PM → 7:27PM)

$1 - \left(\frac{1}{2}\right)^N$

Note: The intuition: it is often easier to calculate the complementary of an event.

The complementary of "at least 1 head" is "no head at all" = all tails

$\mathbb{P}[\text{"no head all"}] = \mathbb{P}[\text{"all tails"}] = \left(\frac{1}{2}\right)^N$

$\mathbb{P}[\text{"at least 1 head"}] = 1 - \mathbb{P}[\text{"no head at all"}] = 1 - \left(\frac{1}{2}\right)^N$

This is a direct proof / explanation, but we could also do a...

PROOF BY INDUCTION.

- DEFINE PREDICATE:

$P(N)$: the probability of having at least 1 head in a sequence of N fair coin flips is $\mathbb{P}[Y] = 1 - \left(\frac{1}{2}\right)^N$

complement $\bar{Y}$: "having no heads in a sequence of N flips"

$\mathbb{P}[\bar{Y}] = 1 - \mathbb{P}[Y]$

- BASE CASE N=1:

Only one coin, requires drawing head → 1/2

$1/2 = 1 - \left(\frac{1}{2}\right)^1$ ✓

P(1) is true.

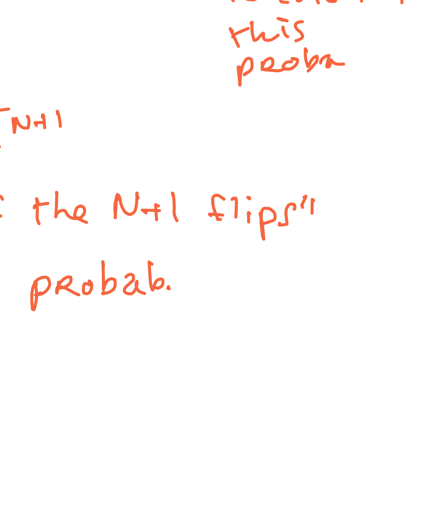
- INDUCTIVE CASE:

1. We assume that P(N) is true ← INDUCTIVE HYPOTHESIS
2. We want to show that P(N+1) is true. ← GOAL
3. consider N+1 coin flips ○ ○ ○ … ○ ○ (N, N+1)
4. you can separate them as the first N flips and the (N+1)th

every step is about calculating ~~the probability~~ that none of the coin flips is head

- then we break into two cases
  - A already a head in N first flips → (N+1) flip doesn't matter
  - B no heads yet, in which case the flip matters
- we want to use this information to calculate the probability that there are no heads (complement of what we are looking for)

B → $1/2^N$ by inductive hypothesis



then the probability that the N+1 coin flip is heads is $\boxed{1/2}$

the overall probability is $1/2^N \cdot 1/2 = \frac{1}{2^{N+1}}$

$\frac{1}{2^{N+1}}$ is the probability of "no heads in any of the N+1 flips"

so the complement "at least 1 flip" has probab.

$1 - \frac{1}{2^{N+1}}$

Therefore P(N+1) is true.

- CONCLUSION:

Since P(1) is true (BASE CASE) and $P(N) \Rightarrow P(N+1)$ (INDUCTIVE CASE)

then P(N) holds for all $N \geq 1$