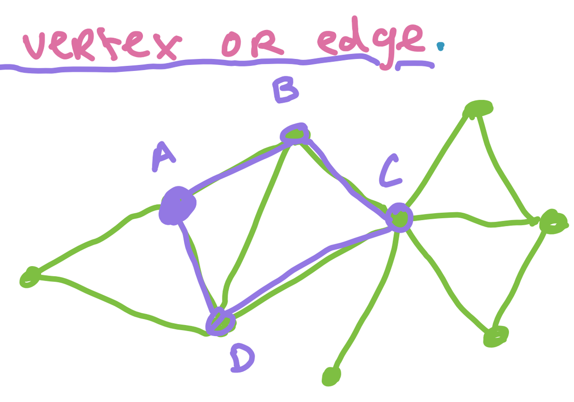
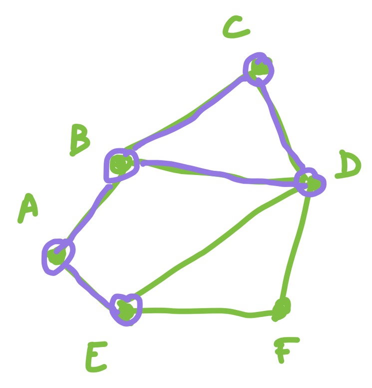
Lecture 25: More Graphs

CYCLE (=rubberband)

A cycle in an undirected graph is a path that starts and ends in the same vertex without repeating any other vertex or edge.





WALK has repeated vertices inside the sequence

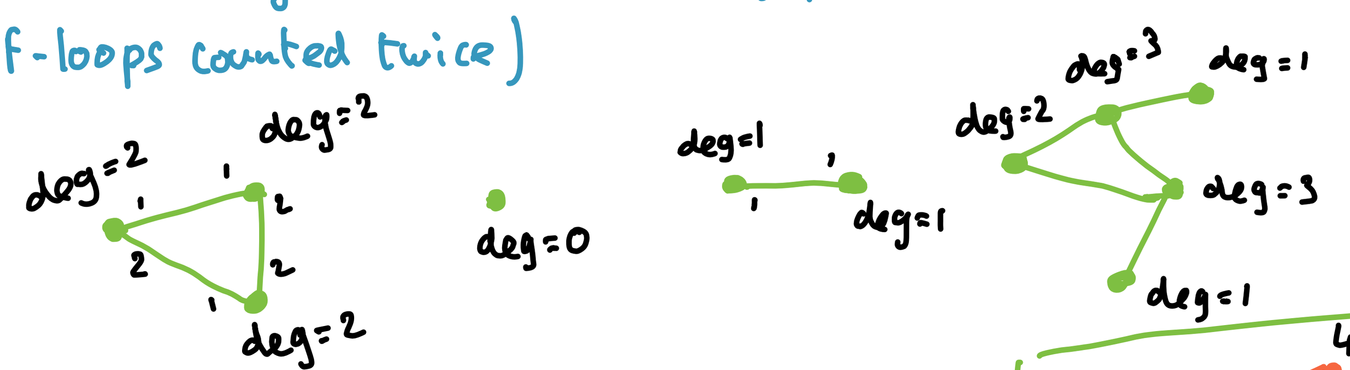
A, B, D, C, B, A, E

(because I am lazy I am not writing the edges in between the vertices) ⚠

a PATH does NOT have repeated vertices

DEGREE

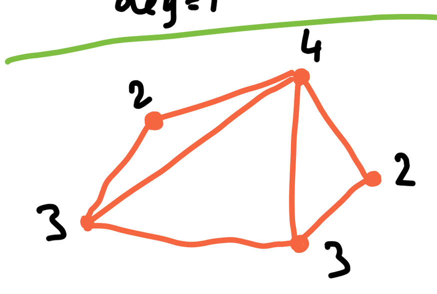
The DEGREE of a VERTEX is the number of edges incident to it (w/ self-loops counted twice)



Thm: In any graph $G=(V,E)$, we have

$$\sum_{v \in V} \deg(v) = 2 \cdot |E|$$

in other words, the sum of degrees is equal to twice the number of edges.



sum of degrees $3+2+4+2+3 = 14$

number of edges 7 =

$2 \cdot |E|$ 14

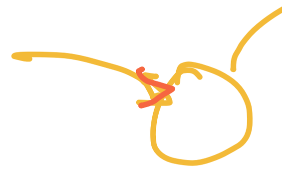
(Handshake Lemma)

Proof (direct proof).

- Each edge in an undirected graph contributes exactly 2 to the total degree count. This is because each edge is connected to two vertices, and contributes 1 to the degree of each.
- SUMMING DEGREES: Consider the sum of the degrees of all vertices in the graph. Each time an edge is counted, it adds 1 to the degree of each of its endpoints.
- TOTAL DEGREE COUNT: Since every edge contributes 2 to the total degree count (1 for each endpoint), the sum of the degrees of all vertices in the graph is twice the number of edges.

Conclusion: Therefore, in any undirected graph the sum of the degrees is equal to twice the number of edges.

in a directed graph



what is the degree?

- undirected
- no self-loops
- no parallel edges

in this course

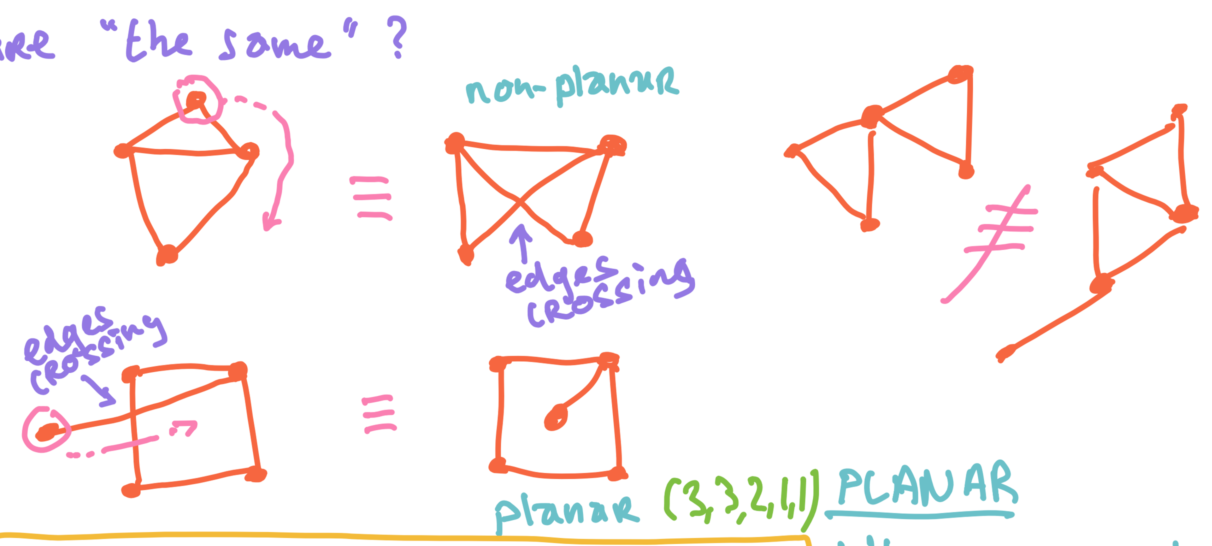
⚠ Go see the ALTERNATE PROOF BY INDUCTION, in the lecture notes

the main idea in the induction is:

- rebuilding a graph of size k given a graph of size $k-1$
- take a graph → pick a vertex → remove the vertex → APPLY INDUCTIVE HYPOTHESIS → reconnect the extra vertex

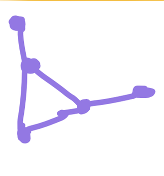
GRAPH ISOMORPHISM

The question of identifying whether two graphs are "the same"?



(3,3,2,1,1) PLANAR

When a graph can be entirely drawn such that no edges cross, then the graph is said to be PLANAR.



Isomorphic to A: R

Isomorphic to S: V, Z, M

Isomorphic to F: T

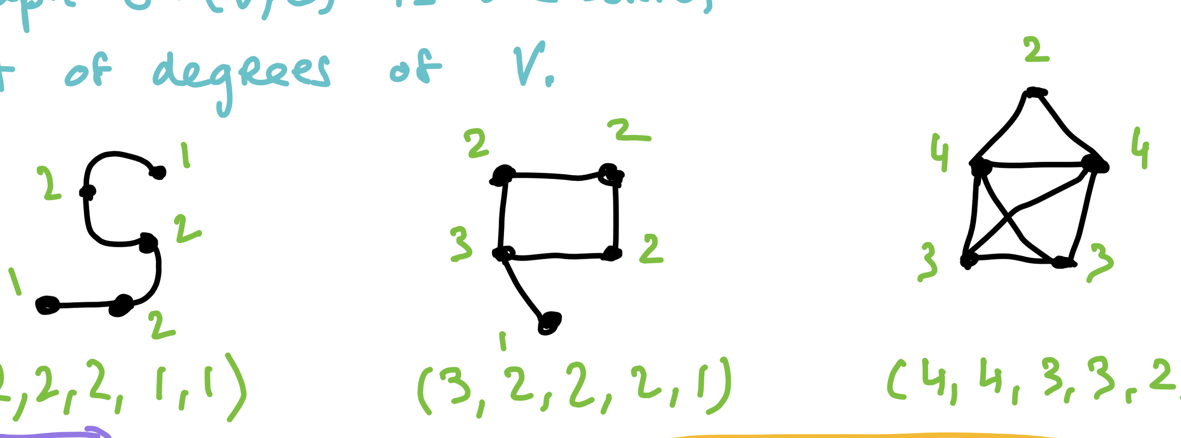
Isomorphic to K: X

RULES

- Same number of vertices?
- Same number of edges?
- Same degree sequence?
- If vertices in one graph can form a cycle of size k can this be formed in the other graph?

DEGREE SEQUENCE

The degree sequence of a graph $G=(V,E)$ is the sorted list of degrees of V.



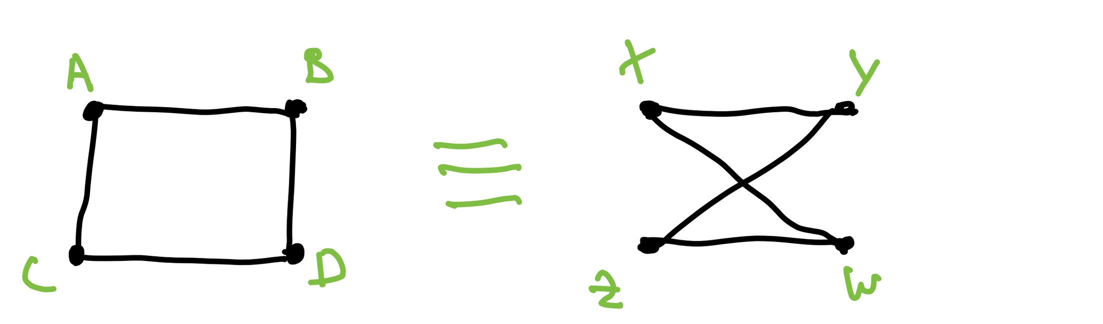
(2,2,2,1,1) (3,2,2,2,1) (4,4,3,3,2)

decreasing

Def Let $G=(V,E)$ and $G'=(V',E')$. G and G' are ISOMORPHIC if there is a bijection $f: V \to V'$, such that for every pair of vertices $(x,y) \in V^2$, $(x,y) \in E$ if and only if $(f(x), f(y)) \in E'$.

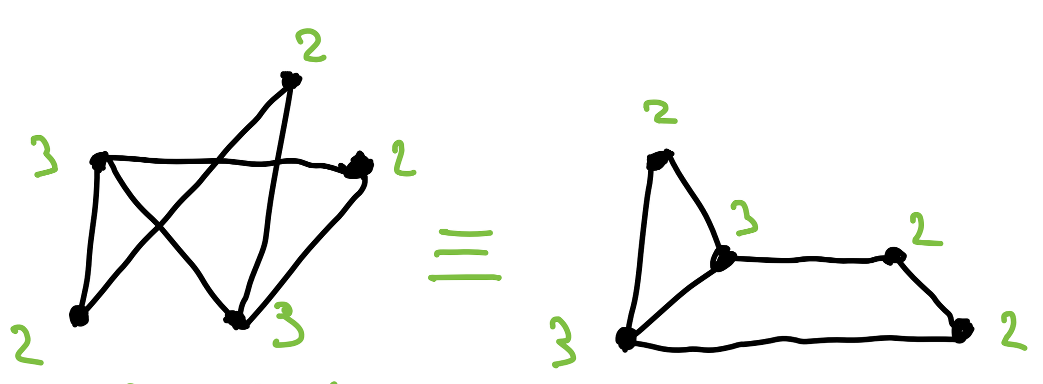
f is a relabeling

Examples



f
A → x
B → y
C → z
D → w

5 vertices
6 edges



5 vertices
6 edges

(3,3,2,2,2) (3,3,2,2,2)