

LECTURE 20: More induction

Note: We did the Induction Team Challenge.

Here is an example statement proved by (weak) induction.

Thm: $\sum_{i=1}^n i^3 = \left(\frac{n(n+1)}{2}\right)^2$

Proof. We proceed by induction. Let

$P(n) : \sum_{i=1}^n i^3 = \left(\frac{n(n+1)}{2}\right)^2$ ← the predicate

BASE CASE. For $n=1$:

LHS: $\sum_{i=1}^1 i^3 = 1^3 = 1$

RHS: $\left(\frac{1(1+1)}{2}\right)^2 = \left(\frac{2}{2}\right)^2 = 1^2 = 1$

Reminder

$f(x) \Big|_{x=y}$

means "I compute the function $f(x)$ when x is set to y ."

The LHS is equal to the RHS. ✓

INDUCTIVE CASE.

We assume that $P(n)$ is true.

[Let's show that therefore $P(n+1)$ is true.]

LHS: $\sum_{i=1}^{n+1} i^3$

$= \sum_{i=1}^{n+1} i^3$

$= \left[\sum_{i=1}^n i^3 + \sum_{i=n+1}^{n+1} i^3 \right]$

$= \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3$

$= \frac{(n(n+1))^2}{2^2} + \frac{2^2 \cdot (n+1)^3}{2^2}$

$= \frac{(n(n+1))^2 + 4(n+1)^3}{2^2}$

$= \frac{n^2(n+1)^2 + 4(n+1)^2(n+1)}{2^2}$

$= \frac{(n+1)^2[n^2 + 4(n+1)]}{2^2}$

$= \frac{(n+1)^2[n^2 + 4n + 4]}{2^2}$

$= \frac{(n+1)^2(n^2 + 4n + 4)}{2^2}$

$= \frac{(n+1)^2(n+2)^2}{2^2}$

$= \left(\frac{(n+1)(n+2)}{2}\right)^2$

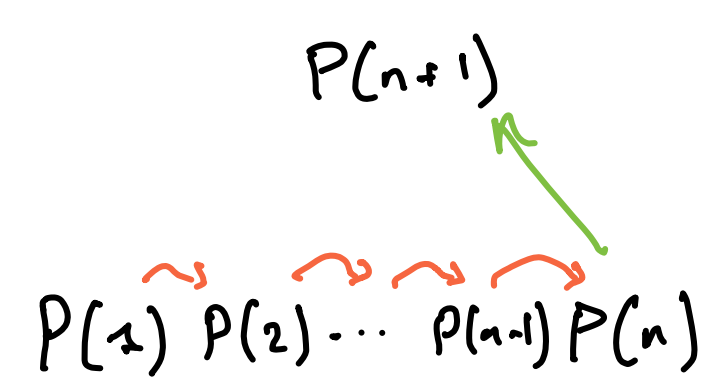
Thus we have shown:

$\sum_{i=1}^{n+1} i^3 = \left(\frac{(n+1)(n+2)}{2}\right)^2$

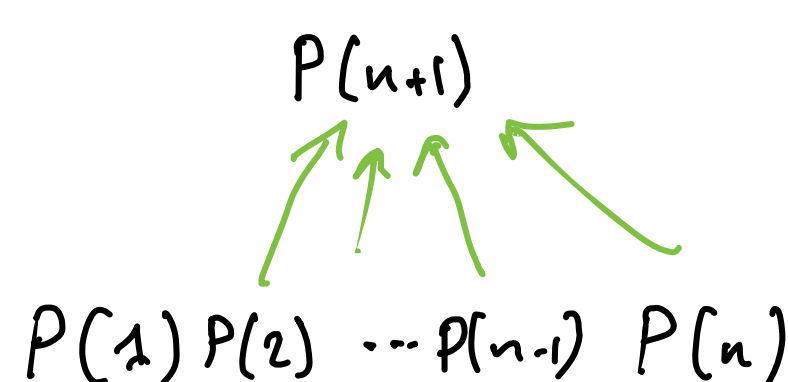
Therefore $P(n+1)$ is true.

CONCLUSION: By induction $P(n)$ has been shown to be true for all integers n , $n \geq 1$.

WEAK INDUCTION vs STRONG INDUCTION



WEAK INDUCTION
only one induction hypothesis



STRONG INDUCTION
more than one induction hypothesis

(confusing part: they are equally powerful but strong idea is sometimes smoother)

Shenao

Diagram illustrating the inductive step for the sum of cubes:

Start: $\sum_{i=1}^n f(i) = f(1) + f(2) + f(3) + \dots + f(n-1) + f(n)$

End: $\sum_{i=1}^{n+1} f(i) = \sum_{i=1}^n f(i) + f(n+1)$

Let's play around with the terms to get intuition:

$(n+1)^3 = (n+1)^2(n+1)$

$= (n^2 + 2n + 1)(n+1)$

We recognize $(n+1)^2$ as a factor in common.

$n^2 + 4n + 4$ is the expansion of $(n+2)^2$

$(a+b)^2 = a^2 + 2ab + b^2$

(this looks weird because the sum only contains one term)

INDUCTIVE STEP
= INDUCTIVE CASE