Lecture 22: Proof Medley — on Probability!

Some tips on when to choose which proof method:

* DIRECT PROOF, to prove $A \Rightarrow B$ when it is straightforward to go from A to B, and A and B are simple to define
* PROOF BY CONTRAPOSITIVE, to prove $A \Rightarrow B$, where either: (i) A is "complicated" or (ii) B is "complicated" but not B is "simple"
* PROOF BY CONTRADICTION, is often used to prove "existence" ("does not exist") or "uniqueness" ("is not unique"), or absence of property
* INDUCTION PROOF, any statement that can be indexed by integers, and is true for all integers after a certain point

Probability Statements

① An event A and its complement $\bar{A}$ cannot both have prob. less than ½.

② The expected value of the sum of a finite number of random variables is equal to the sum of their expected value. $[E[A+B] = E[A] + E[B]]$ linearity of expectation

③ The probability of a complement of an event, $\bar{E}$, is $1 - P(E)$.

④ If the probability of an event is not equal to 1 minus the prob. of the complement of the event, then the event is not WELL-DEFINED

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PROOF #2 by INDUCTION

predicate

$P(n)$: given n random variables, X_i, for $1 \le i \le n$,

$$\mathbb{E}[X_1 + \dots + X_n] = \mathbb{E}[X_1] + \dots + \mathbb{E}[X_n].$$

BASE CASE $n=1$: $P(1)$ is true, by def. of the expected value: $\mathbb{E}[X_1] = \mathbb{E}[X_1]$ ✓

INDUCTIVE CASE: Suppose $P(k)$ is true (induction hypothesis). Let's prove $P(k+1)$.

all equivalent

$P(k) \to P(k+1)$

$P(n) \to P(n+1)$

$P(k-1) \to P(k)$

$$\mathbb{E}[X_1 + \dots + X_{k-1} + X_k]$$

$$= \mathbb{E}[(X_1 + \dots + X_{k-1}) + (X_k)] \quad \text{(rearranging terms)}$$

$$= \underbrace{\mathbb{E}[(X_1 + \dots + X_{k-1})]} + \mathbb{E}[(X_k)] \quad \text{(linearity of exp.)}$$

$$= \mathbb{E}[X_1] + \dots + \mathbb{E}[X_{k-1}] + \mathbb{E}[X_k] \quad \text{(by applying the inductive hyp. } P(k)\text{)}$$

This equality is what we wanted to prove, so $P(k+1)$ is true, and $P(k) \Rightarrow P(k+1)$ is true too.

CONCLUSION: [Because the base case is verified, and the inductive step is true] then by induction, $P(n)$ is true for all n.