Lecture 19: More induction with some extras

COMBINATORIAL PROOF

This is any proof that involves a counting argument. In practice this can be used whenever manipulating discrete objects (sets, relations ...)

TEMPLATE 9 (from Schneiermann)
To prove an equation of the form $\boxed{LHS = RHS}$
- Pose a question of the form "In how many ways ...?"
- On the one hand, explain how the LHS answers the question.
- Separately, explain the RHS
- This will lead to LHS = RHS.

Example: $\boxed{2^0 + 2^1 + \dots + 2^{N-1} = 2^N - 1}$? $(n \in \mathbb{N})$

RHS: the number of non-empty subsets of $\{1 \dots N\}$

LHS: for $k = 1$ to N: the number of non-empty subsets with k as its largest number.

RHS: I can explain it this way:
- Let S be a set containing $|S| = n$ elements.
- The number of non-empty subsets is $2^n - 1$, because
  - (first option) we know that the powerset $\mathcal{P}(S)$ has cardinality 2^n and we remove one subset, the empty set, hence $2^n - 1$
  - (second option) for each of the n elements we have a choice to make — to take it or not — and we must take out the empty set (where chose not to take any items).
  - ...

Question
$S = \{1, 2, 3\}$
$\mathcal{P}(S)$ ← powerset
$|S| = 3$ ← cardinality
$|\mathcal{P}(S)| = 2^3 = 8 = 2^{|S|}$
if $|S| = n$ then $|\mathcal{P}(S)| = 2^n$

1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, 5096...

LHS:

LARGEST ELEMENT	SUBSETS of elements 1 ... n with the largest element	
1	$\{1\}$	$1 = 2^0$
2	$\{1, 2\}$ ~~$\{1\}$~~ $\{2\}$	$2 = 2^1$
3	$\{1, 3\}$ $\{3\}$ $\{1, 2, 3\}$ $\{2, 3\}$	$4 = 2^2$
4	$\{1,2,3,4\}$ $\{1, 4\}$ $\{2,4\}$ $\{3,4\}$ $\{1,2,4\}$ $\{1,3,4\}$ $\{2,3,4\}$ $\{4\}$	$8 = 2^3$

does not contain 2

We consider the number of subsets of $\{1, 2 \dots N\}$ whose largest element is some k, $1 \leq k \leq N$. Such subsets must be of the form $\{\dots, k\}$ where the other elements are chosen from $\{1, 2 \dots k-1\}$. We know there are 2^{k-1} ways of making that choice. Since we are considering 1 through N as largest element, when we sum, we have:

$$2^0 + 2^1 + \dots + 2^{N-1}$$

subsets of which 1 is the largest element ... subsets of which N is the largest element

PARTITIONED SUBSETS

Since answers for LHS and RHS are both correct solutions to the same counting problem, we have proven:

$$2^0 + 2^1 + \dots + 2^{N-1} = 2^N - 1.$$

STRONG INDUCTION

Up until now, we have always done "weak" induction. We therefore have: (a) 1 (or more) base cases (b) we have an induction step of the form $P(n) \Rightarrow P(n+1)$
we only need to make one induction hypothesis

But sometimes it is useful or convenient to assume several inductive hypotheses.

formal variables (what matters is the difference/delta between)
$P(n) \Rightarrow P(n+1)$
$P(n-1) \Rightarrow P(n)$
$P(n+7) \Rightarrow P(n+8)$

Let $P(n)$ be a predicate parameterized by a natural number n.
Let $a \leq b$, be two integers. Then $P(n)$ is true if:
- $P(a), P(a+1), P(a+2) \dots P(b-1)$ is true (BASE)
- For all $k \geq b-1$, if $P(j)$ is true for all $a \leq j < k$ then $P(k)$ is also true

may be multiple hypotheses

Then $P(k)$ holds for all $k > b$.

⚠ Weak and strong induction are exactly as powerful. Choosing one over the other is just a matter of convenience!

Example: FUNDAMENTAL THEOREM OF ARITHMETIC

Every integer larger than 1 can be written as a unique product of prime numbers (up to the order of the factors).

$2 = 2^1$ $\quad 6 = 2^1 \times 3^1$ $[= 3^1 \times 2^1]$ $\quad 24 = 2^3 \times 3^1 = 8 \times 3$ ← unique $[= 2 \times 3 \times 2 \times 2 \dots]$

Proof. $P(n)$: n can be written as a [unique] product of prime numbers

BASE: $N = 2$. The statement $P(2)$ is trivially true because 2 is a prime number, and thus is already expressed as a "product of primes".

INDUCTION: