LECTURE 17

INDUCTION

Recognizing a pattern

What pattern are we seeing?

$1+3 = 4 = 2^2$

$1+3+5 = 9 = 3^2$

$1+3+5+7 = 16 = 4^2$

~~prime numbers?~~
odd numbers?

$1+3+5+7+9 = 25 = \boxed{5}^2$

what are we seeing about odd numbers?

$9 = 2\times\boxed{5}-1$

(Sum of odd Numbers up until N) $= N^2$???

P(N): $\boxed{\sum_{i=1}^{N} 2i-1 = N^2}$???

Idea behind INDUCTION

Induction is used to prove a theorem holds for all integers $(n \geq 0)$ if we are able to do the following:

* state the proposition as depending on an integer parameter P(N)
* show the BASE CASE, P(k) is verified for k=1 (or k equals a small number)
* show that if theorem holds for k then it must also k+1 (INDUCTIVE STEP)

⚠ the base case must exist but is not always 0 or 1

⚠ $K \to K+1$ / $K-1 \to K$ } same thing

FORMAL LETTERS

when we have all three elements we can prove that P(N) holds for all $N \geq k$ for a certain K

Intuition

* the INDUCTIVE STEP says that
  if P(k) then P(k+1)
  that means
  if P(1) then P(2)
  but if P(2) then P(3)
  but if P(3) then P(4)
  but if P(4) ...

BASE CASE

Let P(k) be the property:

$$\boxed{P(k): \sum_{i=1}^{k} 2i-1 = k^2}$$

property / statement that depends on an integer k

BASE CASE:

⚠ We have to identify which value of k makes sense to begin with. (Usually this k=0 or k=1)

(Let's look at k=1)

For k=1:

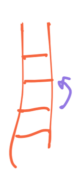
LHS $\sum_{i=1}^{1} 2i-1 = 2\times1-1 = 1$

RHS $1^2 = 1$

LHS = RHS $\Rightarrow$ P(1) ✓

BASE CASE

when do we use cases?
↳ when all possibilities are possible
≠ mutually exclusive situation



one rung of the ladder to the next

get on the ladder

picking the wrong case is like jumping to catch but not actually catching it

INDUCTIVE STEP

We assume P(k) is true. We want to show that P(k+1) is also true.

P(k) is true $\Rightarrow \sum_{i=1}^{k} 2i-1 = k^2$ ← MY SECRET WEAPON

Let's see what that means for $\boxed{\sum_{i=1}^{k+1} 2i-1}$

RECIPE TO FIND A BASE CASE
- try 1,
- if that doesn't work, try 0
- if that doesn't work investigate other options

$$\sum_{i=1}^{k+1} 2i-1 = \left[2(k+1)-1\right] + \underbrace{\sum_{i=1}^{k} 2i-1}_{k^2}$$

Let's Remove the $(k+1)^{th}$ term from the sum

$= 2(k+1)-1+k^2$

Here we want to find a way to turn this expression into $(k+1)^2 = k^2+2k+1$

$= 2k+2-1+k^2$

$= k^2+2k+1$

$= (k+1)^2$ ✓ P(k+1) is true

Therefore we have shown that

P(k) ⟹ P(k+1)

Since BASE CASE + INDUCTIVE STEP then we have shown that P(n) is true for all $n \geq 1$.

$\boxed{\sum_{i=1}^{n} 2i-1 = n^2}$ is true for all $n \geq 1$

Example with BASE CASE different than K=0 or K=1

Question: Is N! bigger or is 2^N bigger?

Exploration

(A) Computing for small values of N

N	N!		2^N
1	1	<	2
2	2	<	4
3	6	<	8
4	24	>	16
5	120	>	32
6	720	>	64

(B) Reasoning abstractly for large values

N terms

$N!$: $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdots$

$\wedge\ \wedge\ \vee\ \vee\ \vee\ \vee\ \vee\ \vee$

2^N: $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdots$

N terms

Things I've uncovered
- N! and 2^N are products of N terms
- all but the first two terms of N! are larger than 2^N
- therefore we expect $\boxed{N! > 2^N,\ N \geq 4}$

It is looking at (A) and (B) that gives me the intuition to
- decide what is the right P(n)
- decide what is the right base case

We want to prove $\boxed{P(N): N! > 2^N}$ is true for all $N \geq 4$.

BASE CASE: Let N=4, then LHS $N! = 4! = 24$, RHS $2^N = 2^4 = 16$

so since $24 > 16$, we see that P(4) ✓ is true

INDUCTIVE STEP: We assume that P(N) is true which means $\boxed{N! > 2^N}$. (my secret weapon!)

We want to know whether this pattern continues with $(N+1)!$ and 2^{N+1}
LHS RHS

$(N+1)! = (N+1)\times N!$ (by def. factorial)

$> (N+1)\times 2^N$ (by inductive hypothesis)

$> 2\times 2^N$ (because $(N+1) > 2$)

$= 2^{N+1}$

$\boxed{(N+1)! > 2^{N+1}}$ ✓ → therefore P(N) ⟹ P(N+1)

Therefore for all $N \geq 4$ we have shown $N! > 2^N$

Addendum on combinatorial proofs

LHS = RHS ?

derive then separately

then show they are equal

(TEMPLATE 9)

What is 'LHS'?

↳ only one derivation

we have mostly done this type of example