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EXERCISE: Show that for $p \geq 5$ and $p \in \mathbb{Z}$, $4p < 2^p$.

1. Define the predicate
(The predicate is a function that maps an positive integer to a logic statement.)

$P_1(p): p \geq 5$ and $p \in \mathbb{Z}$, $4p < 2^p$ ← ok, good
$\boxed{P_2(p): 4p < 2^p}$ ← good enough
$P_3(p): p \in \mathbb{Z}$ and $(p < 5$ or $4p < 2^p)$ ← perfect but will make things harder)

(There are several ways to define the predicate.)

$P_2(0)$: false
$P_2(1)$ = false
$P_2(2)$ = false
$P_2(3)$ = false
$P_2(4)$ = false
$P_2(5)$ = true
⋮
true

$P_3(0)$: true
$P_3(1)$ = true
$P_3(2)$ = true
$P_3(3)$ = true
$P_3(4)$ = true
$P_3(5)$ = true
⋮
true

2. Base case (we look for the first value of p for which P(p) is true)

$P(p): 4p < 2^p$

$p=5$
Now we check that it is true.
The typical approach is either:
* Start from LHS, and simplify. Then simplify RHS and show they are equal.
* Start either from the LHS or the RHS and transform the expression.

LHS: $4p \big|_{p=5} = 4 \times 5 = 20$
RHS: $2^p \big|_{p=5} = 2^5 = 32$

$20 < 32$
therefore
LHS < RHS
therefore

$4 \times 5 < 2^5$

So P(5) is true and we have shown the base case

3. Inductive step
(INDUCTIVE HYPOTHESIS)
"I suppose it's true and let's see what happens"
Let's assume that P(p) is true. optional
[Let's show that this implies P(n+1).]
Because P(p), we have

$4p < 2^p$

Consider P(p+1): [we need to do algebraic manipulation to end up on a term that resembles the induction hypothesis]
LHS: $4(p+1) = 4p + 4$
RHS: $2^{p+1} = 2^p \cdot 2$

We can divide both terms by the same positive factor without changing the relative order of the terms.
LHS: $(4p+4)/2 = 2p + \boxed{2}$ (1) needs some "transforming"
RHS: $(2^p \cdot 2)/2 = 2^p$ (2) (looks like the term in P(p)!)

Because $p \geq 5$ then:
$\boxed{2 < 2p}$. (3)
I'm picking 2p because it will help me get closer to expressing 2p+2 in terms of 4p

By combining (1) (2) and (3), I get
$2p+2 < 2p+2p = 4p$
By applying the IH,
$4p < 2^p$.

Let's rewind! For P(p+1) we have to show that $\underbrace{4(p+1)}_{LHS} < \underbrace{2^{p+1}}_{RHS}$

(I can divide by a positive without changing the relative of the terms)

because $p \geq 5$

$\frac{LHS}{2} = [4(p+1)]/2 = [4p+4]/2 = 2p+2 < 4p$

$\frac{RHS}{2} = [2^{p+1}]/2 = [2^p \cdot 2]/2 = 2^p$

By our previous inductive hypothesis, P(p), we have $4p < 2^p$, therefore we can conclude

$\frac{LHS}{2} < \frac{RHS}{2}$

therefore

LHS < RHS

Therefore P(p+1) holds.

4. By induction P(p) is true for all $p \geq 5$.

Proof of $A \cup B = B \cup A$

1. $A \cup B \subseteq B \cup A$
Let $x \in A \cup B$. Then by def. of union either $x \in A$ or $x \in B$.
If $x \in A$, then by def. of union $x \in B \cup A$.
If $x \in B$, then by def. of union, $x \in B \cup A$. ✓

2. $B \cup A \subseteq A \cup B$
copy-paste (almost)

- To prove equality of sets like $X = Y$ you do $X \subseteq Y$ and $Y \subseteq X$.
- To show $X \subseteq Y$, you start by assuming $x \in X$ and you show that $x \in Y$.

Other proof of $A \cup B = B \cup A$
By commutativity of set union,
$A \cup B = B \cup A$.