LECTURE 20: More induction

Note: We did the Induction Team Challenge.

Here is an example statement proved by (weak) induction.

Thm: $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$.

Proof. We proceed by induction. Let

$$\boxed{P(n) : \underbrace{\sum_{i=1}^{n} i^3}_{LHS} = \underbrace{\left(\frac{n(n+1)}{2}\right)^2}_{RHS}.}$$ ← the predicate

BASE CASE. For $n=1$:

LHS: $\sum_{i=1}^{n} i^3 \Big|_{n=1} = \sum_{i=1}^{1} i^3 = 1^3 = 1$

RHS: $\left(\frac{n(n+1)}{2}\right)^2 \Big|_{n=1} = \left(\frac{1(1+1)}{2}\right)^2 = \left(\frac{2}{2}\right)^2 = 1^2 = 1$

The LHS is equal to the RHS. ✓

Reminder: $f(x)\big|_{x=y}$ means "I compute the function $f(x)$ when x is set to y."

INDUCTIVE STEP = INDUCTIVE CASE

INDUCTIVE CASE.

We assume that $P(n)$ is true. ← INDUCTIVE HYPOTHESIS

[Let's show that therefore $P(n+1)$ is true.]

LHS: $\sum_{i=1}^{n} i^3 \Big|_{n=n+1} = \sum_{i=1}^{n+1} i^3$

$= \left[\sum_{i=1}^{n} i^3 + \sum_{i=n+1}^{n+1} i^3\right]$ SPLIT THE SUM

$= \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3$ APPLY THE INDUCTIVE HYPOTHESIS

$= \frac{(n(n+1))^2}{2^2} + \frac{2^2 \times (n+1)^3}{2^2}$ ALGEBRAIC MANIPULATIONS

$= \frac{(n(n+1))^2 + 4(n+1)^3}{2^2}$ set to common denominator

$= \frac{n^2(n+1)^2 + 4(n+1)^2(n+1)}{2^2}$

$= \frac{(n+1)^2\left[n^2 + 4(n+1)\right]}{2^2}$

$= \frac{(n+1)^2\left[n^2 + 4n + 4\right]}{2^2}$

$= \frac{(n+1)^2 (n+2)^2}{2^2}$

$= \left(\frac{(n+1)(n+2)}{2}\right)^2$

let's play around with the term to get intuition:

$(n+1)^3 = (n+1)^2(n+1)$

$= (n^2 + 2n + 1)(n+1)$

we recognize $(n+1)^2$ as a factor in common

$\boxed{n^2 + 4n + 4}$ is the expansion of $(n+2)^2$

$(a+b)^2 = a^2 + 2ab + b^2$

$\boxed{f(i) = i^3}$

$\sum_{i=1}^{n} f(i) = f(1) + f(2) + f(3) + f(4) + \dots + f(n-1) + f(n)$

start: $(i=1)$, $(i=2)$, $(i=3)$, $(i=4)$, …, $(i=n-1)$, $(i=n)$ end

$\sum_{i=2}^{4} f(i)$

$\sum_{i=1}^{n-1} f(i)$

$\sum_{i=n}^{n} f(i)$ (this looks weird because the sum only contains one term)

$n \to n+1$

$k \to k+1$

$n+1 \to n+2$

$n-1 \to n$

Thus we have shown:

$$\boxed{\sum_{i=1}^{n+1} i^3 = \left(\frac{(n+1)(n+2)}{2}\right)^2}$$

Therefore $P(n+1)$ is true.

CONCLUSION: By induction $P(n)$ has been shown to be true for all integers n, $n \geq 1$.

WEAK INDUCTION vs STRONG INDUCTION

$P(1)\ P(2) \cdots P(n-1)\ P(n) \to P(n+1)$

WEAK INDUCTION
only one induction hypothesis

$P(1)\ P(2) \cdots P(n-1)\ P(n) \to P(n+1)$

STRONG INDUCTION
more than one induction hypothesis

(confusing part: they are equally powerful but strong idea is sometimes smoother)

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