Lecture 18

A different kind of induction proof

$\boxed{P(n) : \text{for all positive integer } n, \quad n^2 < n}$

Let's prove that P(n) is true (or let's try)...

INDUCTIVE HYPOTHESIS: We assume that the property holds for n, P(n) is true.

WILL REVISE

We want to show that from that, P(n+1) is true.

$(n+1)^2 = n^2 + 2n + 1$ (by algebra)

$(n+1)^2 < n + 2n + 1$ (by inductive hypothesis P(n))

$(n+1)^2 < 3n + 1$

$(n+1)^2 < 3n + 1$

$(n+1)^2 < n + 1$

We conclude that P(n+1) is true and therefore

Factorial algorithm

Factorial is a function defined mathematically by

$$\boxed{n! = \begin{cases} 1 & \text{if } n = 0 \\ n * (n-1)! & \text{if } n > 0 \end{cases}}$$

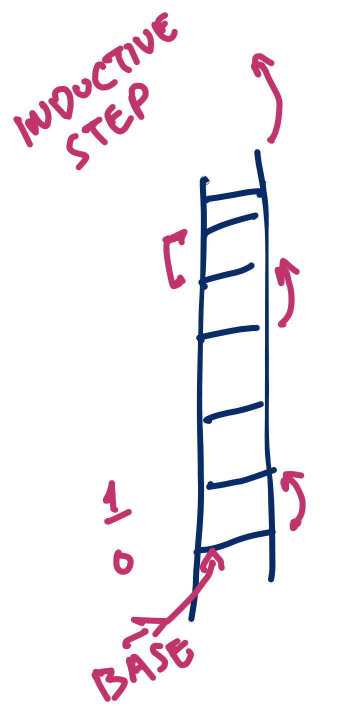
RECURSIVE FUNCTION

Contract

$\boxed{P(n): \text{FACT}(n) = n!}$

In Java:

```
int FACT (int n) {
    if (n == 0) return 1;
    else return n * FACT (n-1);
}
```



We want to prove Factorial(n) computes n!

BASE CASE: The base case is n=0. P(0) is true

By definition of factorial, 0! = 1. ✓

By definition of FACT, FACT(0), the function returns 1 because n==0 is true. ✓

This shows that FACT(0) = 0!, as we want to show.

INDUCTIVE STEP: Let's assume our property is true for n, $\boxed{P(n) \text{ is true}}$, we want to show it is ~~therefore~~ true for n+1 (P(n+1))

IH

By the algorithm, when you call FACT(n+1), the algorithm computes (n+1) * FACT((n+1) - 1)

FACT(n)

By inductive hypothesis, we ~~know~~ that FACT(n) = n!, so we can substitute:

$(n+1) * \text{FACT}((n+1)-1) = (n+1) * \text{FACT}(n)$

$= (n+1) * n!$

$= (n+1)!$

We have shown that $\boxed{P(n) \Rightarrow P(n+1)}$ and combined with the base case, this shows P(n) holds for all n.

Another example of algorithms that can be proved induction: MERGESORT, HEAPSORT

Some birds fly

$\boxed{\exists x \ \text{Bird}(x) \wedge \text{Fly}(x)}$

predicates

Some things are not birds

$\boxed{\exists x \ \neg \text{Bird}(x)}$

① Sum of first N integers

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

②

$$\sum_{i=1}^{n} (2i - 1) = n^2$$

③

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

④

$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

⑤

$$\sum_{i=0} R^i = \frac{1 - R^{N+1}}{1 - R}$$

For groups of 5