LECTURE 16

Example of direct proof

Sum of N first integers

$$\sum_{i=1}^{N} i = \underbrace{1+2+3+\dots+N}_{N \text{ terms}} = \frac{N(N+1)}{2}$$

Proof. Let $S = \sum_{i=1}^{N} i$ then $2S = \overbrace{(N+1)+\dots+(N+1)}^{N \text{ times}}$

so $S = N\frac{N+1}{2}$

this is a direct proof: we start our initial object $(S = \sum_{i=1}^{N} i)$ and by making derivations we obtain the final result.

Indirect proof

Main types of indirect proofs:

* proof by contradiction → $(A \wedge \neg B) \to \text{FALSE}$
* proof by contrapositive → $\neg B \to \neg A$

these proofs are called "indirect" because we set out to prove something different than our goal

This material is covered in the SCHEINERMANN chapters on Canvas

If... then... $P \Rightarrow Q$

PIGEONHOLE PRINCIPLE

Statement: If [N pigeons are placed into M pigeonholes with $N > M$]^{A}, then [at least one pigeonhole must contain more than one pigeon.]^{B} $\boxed{A \Rightarrow B}$

To prove $\boxed{A \Rightarrow B}$

You can:

- prove $\neg B \Rightarrow \neg A$ (contrapositive has same truth value as implication)
- Assume $A \wedge \neg B$ and show contradiction

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Proof by contradiction $(A \wedge \neg B) \to \text{FALSE}$

$\neg B$: every pigeonhole contains at most one pigeon

1. **Assume the opposite:** Suppose that N pigeons are placed into M pigeonholes with $N > M$ and every pigeonhole contains at most one pigeon.
2. This means the maximum number of pigeons that can be placed without any pigeonhole containing more than one pigeon is M.
3. **Build-up to contradiction:** But we have N pigeons and $N > M$. This contradicts our original assumption that every pigeonhole contains at most one pigeon.
4. **Conclude that $\neg B$ can't be true:** Therefore our assumption $\neg B$ is false and hence the Pigeonhole Principle is true. ∎

Proof by contrapositive $(\neg B \Rightarrow \neg A)$

Statement: If [N pigeons are placed into M pigeonholes with $N > M$]^{A}, then [at least one pigeonhole must contain more than one pigeon.]^{B} $\boxed{A \Rightarrow B}$

Contrapositive:

If [no pigeonhole contains more than one pigeon]$^{\neg B}$ then [$N \leq M$]$^{\neg A}$.

1. Assume that no pigeonhole contains more than one pigeon $(\neg B)$.
2. This means the maximum number of pigeons that can be placed without any pigeonhole containing more than one pigeon is M.
3. Therefore, the number of pigeons (N) must be less than (or equal) to M.
4. This proves the contrapositive $(\neg B \Rightarrow \neg A)$ and hence the original statement ∎

$A \setminus B$ ↳ def. $x \in A$ AND $x \notin B$

after the first step of a proof is unwinding the defs

Application: Can be used to prove

- Injective functions: Let $f: A \to B$ be a function where A and B are finite sets and $|A| > |B|$. Then f cannot be injective (one-to-one)

  ↳ proof using Pigeonhole Principle

Another example of proof by contradiction

Statement. $\sqrt{2}$ is not rational.

Def A number x is RATIONAL iff there exists $p, q \in \mathbb{Z}^2$ such that $x = \frac{p}{q}$ $\quad q \neq 0$ (the Jam) + p and q coprime

Proof (by contradiction).

1. **Assume contradiction.** Let's assume $\sqrt{2}$ is rational.
2. This means that there exists $p, q \in \mathbb{Z}^2$ ($= \mathbb{Z} \times \mathbb{Z}$) with $q \neq 0$ such that $\sqrt{2} = \frac{p}{q}$
3. **Derive until you get a contradiction:**

$$\sqrt{2} = \frac{p}{q}$$

squaring both sides

$$2 = \frac{p^2}{q^2}$$

(✱) $2q^2 = p^2$

This means that $\boxed{p^2 \text{ is even}}$ (since divisible by 2)

Consequence: $\boxed{p \text{ is even}}$ ①

4. Therefore there exist $k \in \mathbb{Z}$ such that $p = 2k$. substituted in (✱)

$$2q^2 = (2k)^2$$

square

$$2q^2 = 4k^2$$

divide by 2

$$\boxed{q^2 = 2k^2}$$

Consequence: $\boxed{q \text{ is even}}$ ②

5. ① + ② If both p and q are even then they share a common factor, they are coprime, which conflicts with our original assumption that p and q are coprime.
6. Therefore our original assumption that $\sqrt{2}$ is rational must be false.
7. Therefore $\sqrt{2}$ is irrational.