CIT 5920
Recitation 12

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Penn Engineering

Overview for Today

- Logistics
- Graph Theory Review
- Question 1: Model Undirected Graph
- Question 2: At Least Two Mutual Friends/Strangers
- Question 3: Prove or Disprove
- Question 4: Handshake Lemma
- Question 5: Real-Life Scenario
- Question 6: Graph Enumeration
- Question 7: Shortest Path & Cycles
- Question 8

Logistics

- Final Exam (In-Person) on **Monday, December 16**
  - **6 PM - 8 PM**
  - **David Rittenhouse Laboratory A8**
- Final exam review session next week (time TBD)
- No HW7 (no homework on graph theory)

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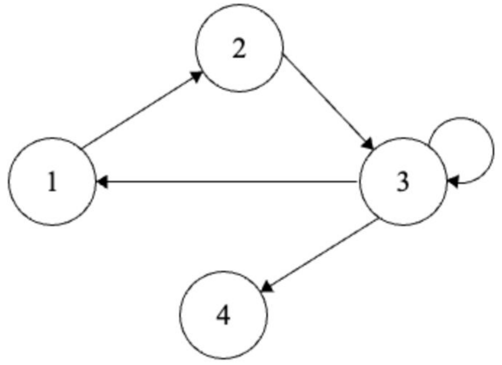
Graph Concepts Review

- We define a graph as G = (V, E)
  - V = {1, 2, 3, 4}, the set of vertices
  - E = {(1, 2), (2, 3), (3, 1), (3, 3), (3, 4)}, the set of edges $E \subseteq V \times V$



- Path: an alternating sequence of vertices and edges
  - Ex. 1 (1, 2) 2 (2, 3) 3 (3, 4) 4
- Cycle: is a path that starts and ends in the same vertex without repeating any other vertex or edge
  - 1 (1, 2) 2 (2, 3) 3 (3,1) 1
- Loop: edge from one vertex to itself (no self loops in this course)
  - (3, 3)
- Undirected: edges have no direction and are represented by pairs {a, b}
- Directed: edges have a direction and are represented as ordered pairs (a, b)
- Degree of a vertex: the number of edges that are incident on it

Graph Theory Review

A graph is **connected** if there is a path between every pair of vertices.

A **connected component** is a maximal subgraph in which any two vertices are connected, and no additional vertices can be included without breaking this property.

A connected acyclic (no cycles) graph is called a **tree**.

Complete graph: a graph where every vertex is connected to every other vertex.
A complete graph has $\binom{n}{2}$ edges.

Graph Theory Review

A **dense** graph is one in which the number of edges is close to the maximum number of possible edges.

A **sparse** graph is one in which the number of edges is much smaller than the maximum possible.

A graph is **bipartite** if its vertices can be divided into two disjoint sets such that no two vertices within the same set are adjacent.

Handshake Lemma

In any undirected graph, the sum of the degrees of all vertices equals twice the number of edges.

Mathematically, if G=(V, E) is a graph with vertex set V and edge set E, then:

$$\sum_{v \in V} \deg(v) = 2|E|$$

The lemma implies that the number of vertices with odd degrees in any undirected graph must be even.

Question 1

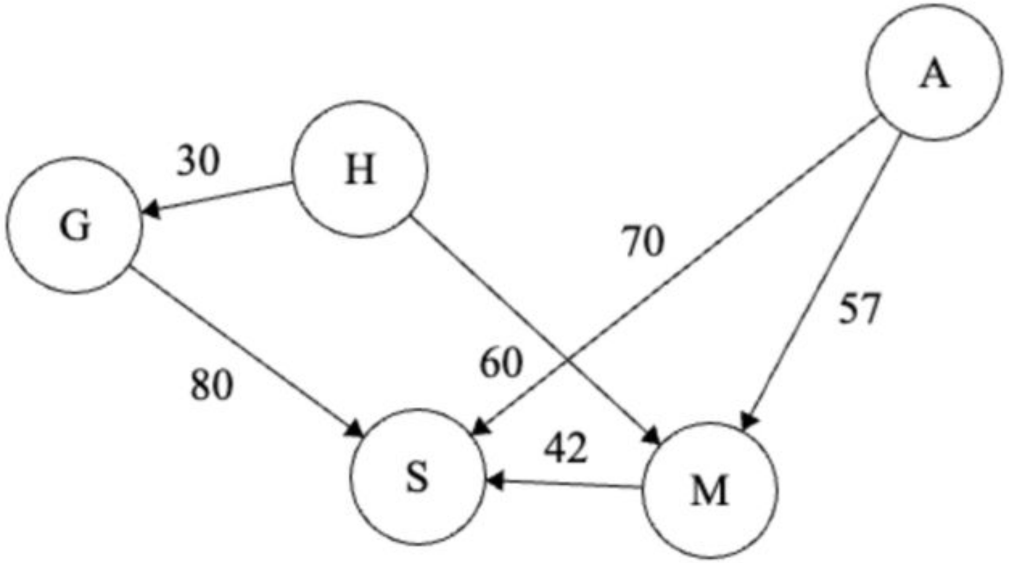
Consider the following fictional cities: Genoa, Hawkins, Aloria, Medford, and Sariola.
There are **two-way** roads built between the following locations with the given lengths:

- Genoa to Sariola, length 80km
- Hawkins to Medford, length 60km
- Medford to Sariola, length 42km
- Aloria to Sariola, length 70km
- Hawkins to Genoa, length 30km
- Aloria to Medford, length 57km

First, model this as an undirected graph with vertices and weighted edges. Explain your choice of vertices and edges, and what the edge weights mean. Why is an undirected graph appropriate here as apposed to a directed graph?
Now what's the shortest distance between Hawkins and Sariola? Between Aloria and Genoa?
Now consider if the roads were **one-way** instead, with the first city listed above as the starting point (e.g. the first road goes *from* Genoa *to* Sariola). How would this change things? How should we model the cities now with a graph?
What are the new shortest distances in the above question, if it's even possible to reach the other city?

Answer 1

Solution: We will make the cities the vertices, and the roads the undirected edges. The weights of the edges represent the distance/length of the roads. Consider the following (directed) graph drawn out:



Initially, considering the undirected version, the shortest distance between Hawkins and Sariola is from Hawkins to Medford, and then Sariola with total distance of $60 + 42 = 102$. From Aloria to Genoa it's 57+60+30 = 147

Now, considering the directed version, the distance between Hawkins and Ariola doesn't change (the roads go in the right direction). However, now it is no longer possible to travel from Aloria to Genoa following the one-way roads.

Question 2

Exercise 1

Show that at a party with at least two people, there are at least two mutual friends or two mutual strangers (assume strangers are the opposite of friends).

By "mutual friends" here, we just mean that A and B are "mutual friends" iff they are adjacent to each other (i.e. A and B are friends).

Same with "mutual enemies."

Answer 2

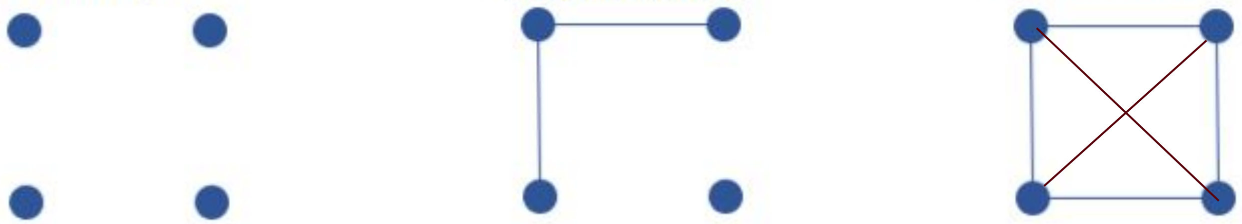
Exercise 1

Show that at a party with at least two people, there are at least two mutual friends or two mutual strangers (assume strangers are the opposite of friends).

> **Solution:** Make a vertex for every person.
>
> Make an edge if 2 people are friends.
>
> Either the graph has an edge or there are no edges at all. If there is an edge we have 2 friends. If there are no edges at all, then we have enemies.

Here are some examples:

All strangers: Some friends: All friends:



Question 3

Exercise 4

Consider a connected undirected graph G with n vertices and m edges. Prove or disprove the following statement:

If every vertex in the graph has a degree of at least 2 (i.e. every vertex has edges connecting it to at least 2 other vertices), then there exists a cycle in G.

Answer 3

Exercise 4

Consider a connected undirected graph G with n vertices and m edges. Prove or disprove the following statement:

If every vertex in the graph has a degree of at least 2 (i.e. every vertex has edges connecting it to at least 2 other vertices), then there exists a cycle in G.

Solution: The statement is true. Let's prove it by contradiction.

Assume, for the sake of contradiction, that there exists a connected undirected graph G with n vertices and m edges where every vertex has a degree of at least 2, but there is no cycle in G.

Since every vertex has a degree of at least 2, there must be at least two edges incident to each vertex. Now, consider starting at any vertex v in G. Since v has a degree of at least 2, there are at least two edges leaving v. Follow one of these edges to the next vertex, and continue this process.

Answer 3

We can follow this process until we got through all the vertices and arrive at the final vertex which we will call x. This vertex must have at least two edges. One of the edges leads to the vertex we visited before this one, but the other edge must lead to a different vertex, let's call it y.

However, since this was the final vertex, y must be a vertex that we have visited before! This means we form a cycle in the graph, which contradicts our assumption that there is no cycle.

Therefore, our initial assumption that there is no cycle in G must be false. Thus, the statement is proven: if every vertex in the graph has a degree of at least 2, then there exists a cycle in G.

(Note: You can also do this proof in almost the same way as a sort of direct proof instead of a proof by contradiction.)

Question 4

Exercise 5
Consider a group of 5 people. A pair of people can either be friends, or not be friends. Assume a person could not be friends with themselves for this question. Is it possible for everyone to be friends with exactly:

A. 0 other people in the group?

B. 1 other person in the group?

C. 2 other people in the group?

D. 3 other people in the group?

E. 4 other people in the group?

Answer 4

Exercise 5
Consider a group of 5 people. A pair of people can either be friends, or not be friends. Assume a person could not be friends with themselves for this question. Is it possible for everyone to be friends with exactly:

A. 0 other people in the group?

Solution: Yes. The graph would simply have no edges.

Answer 4 (cont.)

Exercise 5
Consider a group of 5 people. A pair of people can either be friends, or not be friends. Assume a person could not be friends with themselves for this question. Is it possible for everyone to be friends with exactly:

B. 1 other person in the group?

Solution: No. If we tried to do this, we would pair people up and then be left with one person who was alone.

Answer 4 (cont.)

Exercise 5

Consider a group of 5 people. A pair of people can either be friends, or not be friends. Assume a person could not be friends with themselves for this question. Is it possible for everyone to be friends with exactly:

C. 2 other people in the group?

Solution: Yes. The graph would look like a pentagon: every vertex would have two connect neighbors, forming one big cycle.

Answer 4 (cont.)

Exercise 5

Consider a group of 5 people. A pair of people can either be friends, or not be friends. Assume a person could not be friends with themselves for this question. Is it possible for everyone to be friends with exactly:

D. 3 other people in the group?

Solution: No. We would need an odd number of vertices that had an odd number of degrees. But the sum of degrees must always be even, so this couldn't work.

Answer 4 (cont.)

Exercise 5

Consider a group of 5 people. A pair of people can either be friends, or not be friends. Assume a person could not be friends with themselves for this question. Is it possible for everyone to be friends with exactly:

E. 4 other people in the group?

Solution: Yes. The graph would be a fully connected graph with each vertex being connected to every other vertex.

Question 5: Real-Life Scenario

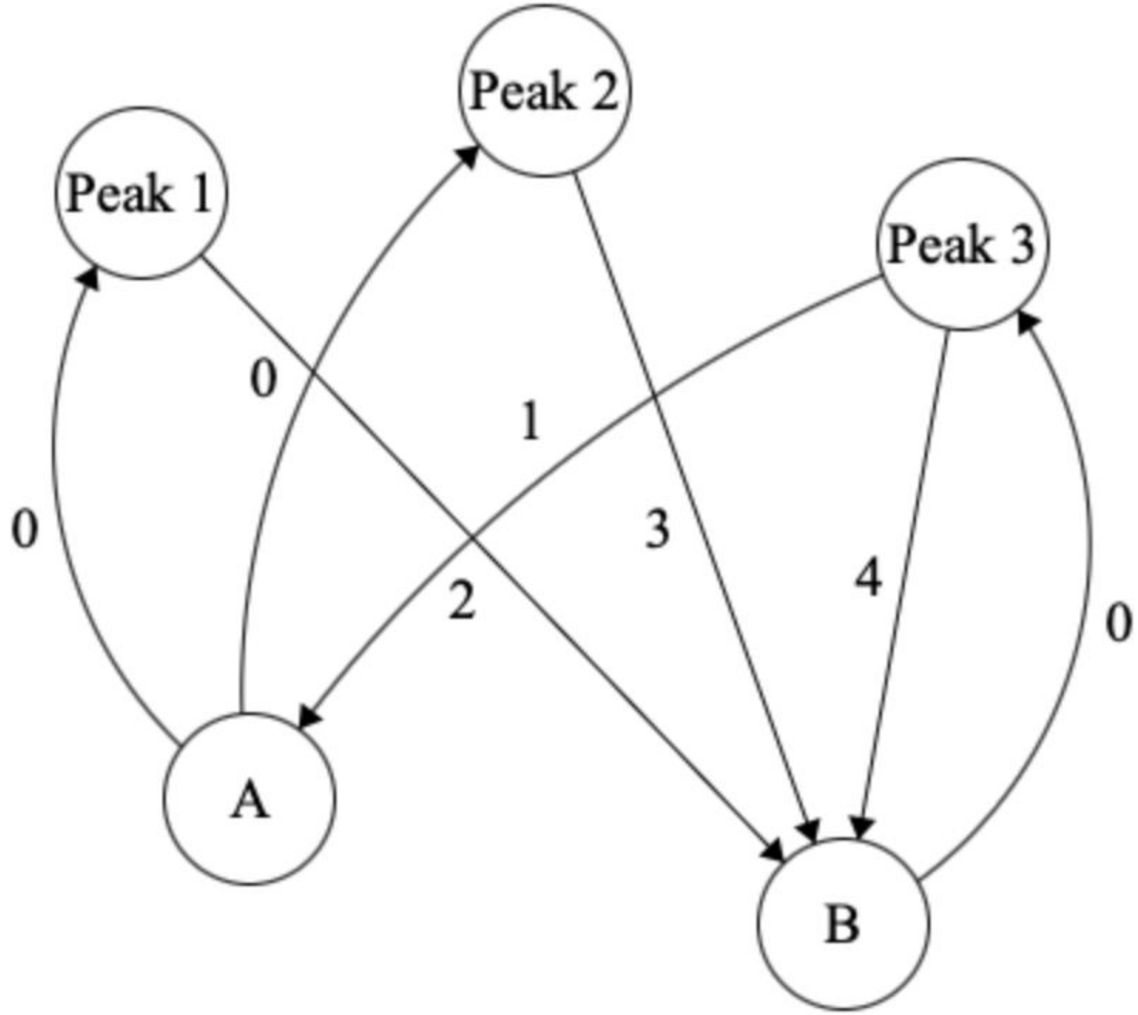
A small ski resort consists of several peaks and lodges, connected by ski lifts and ski runs. The details of the resort are as follows:

1. **(Peaks and Lodges):**
    - Peak 1, Peak 2, Peak 3
    - Lodge A and Lodge B
2. **(Lifts and Pistes):**
    - Lift connections:
        - Lodge A to Peak 1
        - Lodge A to Peak 2
        - Lodge B to Peak 3
    - ski run connections:
        - Peak 1 to Lodge B (blue)
        - Peak 2 to Lodge B (black)
        - Peak 3 to Lodge A (green)
        - Peak 3 to Lodge B (double black)

Task: Represent this scenario as a weighted directed graph.

Answer 5

1. Treat the lodges and peaks like vertices
2. Give the lifts a weight of 0, green 1, blue 2, black 3 and double black 4
3. Connect appropriate edges



Question 6: Graph Enumeration

Exercise 3 – Graph Enumeration

How many directed graphs, no self loops and no parallel edges, can be formed with vertex set $\{v_1, v_2, \ldots, v_n\}$. The vertices are labeled. On a 3 vertex case, we would actually consider $\{(1,2),(2,3)\}$ to be different from $\{(2,1),(3,2)\}$.

Answer 6

Solution: Since the graph is directed, any ordered pair of distinct vertices (v_i, v_j) can either have a directed edge from v_i to v_j or not. We do not consider self-loops, so no vertex can have an edge starting and ending at itself. There are $n(n-1)$ ways to choose an ordered pair of distinct vertices from n vertices.

For each of the $n(n-1)$ pairs, there are two possibilities: Either there is a directed edge between the pair, or there is not.

Therefore, for each of the $n(n-1)$ pairs, we have 2 choices (directed edge or no directed edge), leading to a total of:

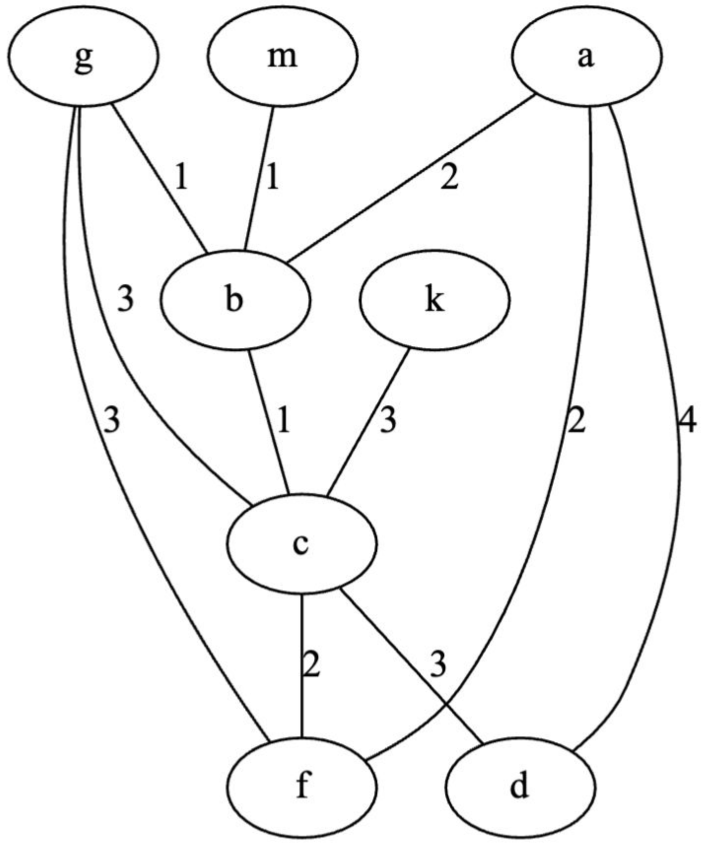
$$2^{n(n-1)}$$

different directed graphs.

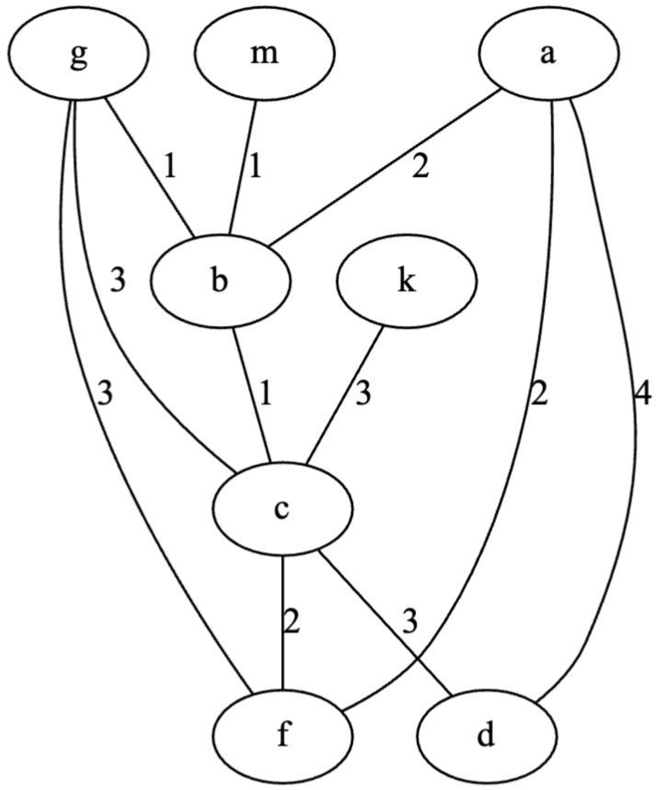
Question 7: Shortest Path & Cycles

What is the shortest path from *a* to *c*?

Does this graph contain any cycles? If so, provide an example.



Answer 7



1. Shortest path: a → b → c, with total edge weight = 2 + 1 = 3

2. Some examples of cycles:
   a. a → b → c → d → a
   b. g → b → c → g
   c. g → c → f → g

Question 8

Translate the following statements into formal mathematical notation:

(i) "The number of ways to choose 3 items from 10 is C."

$$C = \binom{10}{3}$$

(ii) "The statement P implies the statement Q."

$$P \implies Q$$

(iii) "The relation R on set A is reflexive."

R is reflexive on $A \to \forall a \in A, (a, a) \in R$

(iv) "The logical negation of proposition P is true."

"$\neg P$ [is true]" or "$\neg P =$ T[rue]"

(v) "There exists an element in set X for which predicate $P(x)$ is true."

$$\exists x \in X, P(x)$$

Question 8

(vi) "For all elements in set X, the predicate $P(x)$ is false."

$$\forall x \in X, \neg P(x)$$

(vii) "The binary relation R on set X is transitive."

$$R \text{ on } X \text{ is transitive} \rightarrow \forall a, b, c \in X, ((a, b) \in R \wedge (b, c) \in R) \implies (a, c) \in R$$

(viii) "The logical conjunction of propositions P and Q is false."

"$\neg(P \wedge Q)$" or "$(P \wedge Q)$ is false"

(ix) "For every vertex x in graph G, there exists an edge connecting it to vertex y."

$$\forall u \in G, \exists e = (u, v) \in G$$

(x) "For every integer x, there exists an integer y such that x is less than y."

$$\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, x < y$$

Good luck on final exam!

We will miss you :)

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