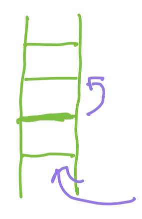
Lecture 23: Induction + Proofs on Graphs

Complete graph: a graph in which every vertex is connected to every other vertex.

K_n : the complete graphs on n vertices
↳ ("Klique" which is "complete graph" in German)

INDUCTION IS LIKE LADDER



STEP 2 (INDUCTIVE STEP)
HOW TO CLIMB TO NEXT RUNG

Step 1 (BASE)
GET ON THE LADDER

STEP 1 + STEP 2 = PROOF

Theorem: A complete graph K_n has $\frac{n(n-1)}{2}$ edges

$$\left[\text{e.g. } \left.\frac{n(n-1)}{2}\right|_{n=5} = \frac{5(5-1)}{2} = \frac{4 \times 5}{2} = 10. \right]$$

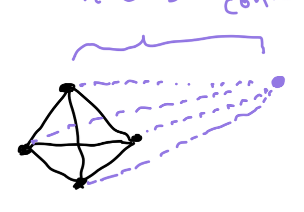
(≈ similar to handshake lemma)

Proof by induction:

- P_k : The complete graph K_k has $\frac{k(k-1)}{2}$ edges. $(k \in \mathbb{N})$ 0. define statement

- BASE CASE: for $k = 1$:
  * A complete graph with 1 vertex has 0 edges. • K_1
  * The formula gives $\frac{1(1-1)}{2} = 0$.
  * The formula matches, therefore P_1 is true.

- INDUCTIVE STEP: Assuming that P_n is true. (Let's see if we can show P_{n+1} to be true.)
  - we have K_n with $\frac{n(n-1)}{2}$ edges.
  - Let's add an additional vertex
  - To connect to all n existing vertices requires n additional edges



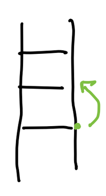
  - Therefore it has

$$\frac{n(n-1)}{2} + n = \frac{n(n-1) + 2n}{2} = \frac{n(n+1)}{2}$$

  - Therefore we have shown that P_{n+1} is true i.e. $P_n \Rightarrow P_{n+1}$.

- CONCLUSION. Since P_1 and $P_n \Rightarrow P_{n+1}$, then P_k for all $k \geq 1$.

SHOW HOW TO GO UP
Assuming K_n to show K_{n+1}

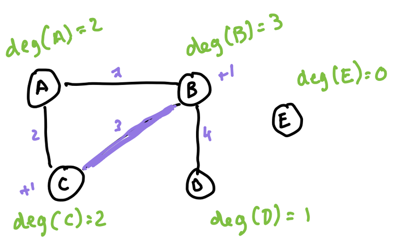


SHOW HOW TO GO DOWN
Assuming K_{n+1} to show K_n

Theorem: In any undirected graph, the sum of degrees of all vertices equals twice the number of edges.

Let $G = (V, E)$ be an undirected graph

$$\sum_{v \in V} \deg(v) = 2|E|$$



$\sum_{v \in V} \deg(v) = \deg(A) + \deg(B) + \deg(C) + \deg(D) + \deg(E)$
$= 2 + 3 + 2 + 1 + 0$
$= 8$ } match!
4 edges $2|E| = 8$

Proof: [DIRECT PROOF]

- Understanding degrees: The degree of a vertex is the number of edges incident/connected to it
- Edge Contribution: Each edge connects to two vertices. Therefore each edge contributes 1 to each of its endpoints
  [Let $(x, y) \in E$ then this adds 1 to $\deg(x)$ and $\deg(y)$]
- Summing the Degrees: When summing the degrees of all vertices, each edge is counted twice (once for each endpoint).
- Conclusion: If the graph has E edges the total sum of the degrees is $2|E|$.

$P_n \Rightarrow P_{n+}$

THREE SUMS TO PROVE BY INDUCTION

PROPERTY	BASE CASE	INDUCTIVE HYPOTHESIS	STRATEGY
LHS (Left Hand Side) $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ RHS (Right Hand Side) sum of first n integers	verify formula holds for $n = 1$ LHS = 1 RHS = $\frac{1(1+1)}{2} = 1$ } equal ✓	Assume the formula holds for N $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$	Add N+1 to both sides and simplify RHS
$\sum_{i=1}^{n} (2i-1) = n^2$ sum of first n odd integers	verify formula holds for $n = 1$ LHS = 1 RHS = $1^2 = 1$ } equal ✓	"	Add the next odd number to both sides and show $(n+1)^2$ is obtained (factorization) $(n+1)^2 = n^2 + 2n + 1$
$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$ sum of geometric series	verify formula holds for $n = 1$ LHS = $2^0 + 2^1 = 1 + 2 = 3$ RHS = $2^{1+1} - 1 = 2^2 - 1 = 4 - 1 = 3$ }	"	Add 2^{N+1} to both sides to show formula

P_k : The sum of the k first integers is $\frac{k(k+1)}{2}$.

STRONG INDUCTION EXAMPLE

Theorem (Fundamental Theorem of Arithmetic)

|| EVERY INTEGER GREATER THAN 1 CAN BE WRITTEN AS A PRODUCT OF PRIME NUMBERS.

Proof by strong induction:

BASE CASE $n=2$: 2 is a prime
It is a product of itself
✓

INDUCTIVE STEP

WRONG SOLUTION using weak induction
- Assume the property is true for N.
- Let's show it is true for N+1
- ✗ STUCK ✗ because we CANNOT CONSTRUCT N+1 AS A PRODUCT OF N

VERTIGO

- Assume every integer m, with $2 \leq m < n$, verifies the property, i.e. can be written as a product of PRIMES] STRONG INDUCTION HYPOTHESIS
- Then we have two cases:
  - Either N is PRIME → done.
  - OR N is not PRIME, it is COMPOSITE
    $N = A \times B$ with $2 \leq A < N$, $2 \leq B < N$ } ⚠ important

    By inductive hypothesis, A and B can both be written as a product of primes
    Therefore N is a product of primes
- Conclusion: BASE + Ind. Step = proof