Proof Templates

From Scheinermann's *Mathematics: A Discrete Introduction*.

Proof Template 1 Direct proof of an if-then theorem.

- Write the first sentence(s) of the proof by restating the hypothesis of the result. Invent suitable notation (e.g., assign letters to stand for variables).
- Write the last sentence(s) of the proof by restating the conclusion of the result.
- Unravel the definitions, working forward from the beginning of the proof and backward from the end of the proof.
- Figure out what you know and what you need. Try to forge a link between the two halves of your argument.

Proof Template 2 Direct proof of an if-and-only-if theorem.

To prove a statement of the form "A iff B":

- ($\Rightarrow$) Prove "If A, then B."
- ($\Leftarrow$) Prove "If B, then A."

Proof Template 3 Refuting a false if-then statement via a counterexample.

To disprove a statement of the form "If A, then B":

Find an instance where A is true but B is false.

Proof Template 4 Truth table proof of logical equivalence

To show that two Boolean expressions are logically equivalent:

Construct a truth table showing the values of the two expressions for all possible values of the variables.

Check to see that the two Boolean expressions always have the same value.

Proof Template 5 Proving two sets are equal.

Let A and B be the sets. To show $A = B$, we have the following template:

- Suppose $x \in A$. ... Therefore $x \in B$.
- Suppose $x \in B$. ... Therefore $x \in A$.

Therefore $A = B$. ∎

Proof Template 6 Proving one set is a subset of another.

To show $A \subseteq B$:

Let $x \in A$. ... Therefore $x \in B$. Therefore $A \subseteq B$. ∎

Proof Template 7 Proving existential statements.

To prove $\exists x \in A$, assertions about x:

Let x be (give an explicit example) ... (Show that x satisfies the assertions.) ... Therefore x satisfies the required assertions. ■

Proof Template 8 Proving universal statements.

To prove $\forall x \in A$, assertions about x:

Let x be any element of A. ... (Show that x satisfies the assertions using only the fact that $x \in A$ and no further assumptions on x.) ... Therefore x satisfies the required assertions. ■

Proof Template 9 Combinatorial proof.

To prove an equation of the form LHS = RHS:

Pose a question of the form, "In how many ways ... ?"
On the one hand, argue why LHS is a correct answer to the question.
On the other hand, argue why RHS is a correct answer.
Therefore LHS = RHS. ■

Proof Template 10 Using inclusion-exclusion.

Counting with inclusion-exclusion:

- Classify the objects as either "good" (the ones you want to count) or "bad" (the ones you don't want to count).
- Decide whether you want to count the good objects directly or to count the bad objects and subtract from the total.
- Cast the counting problem as the size of a union of sets. Each set describes one way the objects might be "good" or "bad."
- Use inclusion-exclusion (Theorem 19.1).

Proof Template 11 Proof by contrapositive

To prove "If A, then B": Assume (not B) and work to prove (not A).

a	b	$a \to b$	$\neg b$	$\neg a$	$(\neg b) \to (\neg a)$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Proof Template 12 Proof by contradiction

To prove "If A, then B":

We assume the conditions in A.
Suppose, for the sake of contradiction, not B.
Argue until we reach a contradiction.
$\Rightarrow\Leftarrow$ ■

(The symbol $\Rightarrow\Leftarrow$ is an abbreviation for the following: Thus we have reached a contradiction. Therefore the supposition (not B) must be false. Hence B is true.)

a	b	$a \to b$	$a \wedge \neg b$	$(a \wedge \neg b) \to$ FALSE
T	T	T	F	T
T	F	F	T	F
F	T	T	F	T
F	F	T	F	T

Proof Template 13 Proving that a set is empty.

To prove a set is empty:
Assume the set is nonempty and argue to a contradiction.

Proof Template 14 Proving uniqueness.

To prove there is at most one object that satisfies conditions:
Proof: Suppose there are two different objects, x and y, that satisfy conditions.
Argue to a contradiction.

Proof Template 15 Proof by smallest counterexample.

First, let x be a smallest counterexample to the result we are trying to prove. It must be clear that there can be such an x.

Second, rule out x being the *very* smallest possibility. This (usually easy) step is called the *basis* step.

Third, consider an instance x' of the result that is "just" smaller than x. Use the fact that the result for x' is true but the result for x is false to reach a contradiction $\Rightarrow\Leftarrow$.

Conclude that the result is true. ■

Proof Template 16 Proof by the Well-Ordering Principle.

To prove a statement about natural numbers:

Proof. Suppose, for the sake of contradiction, that the statement is false. Let $X \subseteq \mathbb{N}$ be the set of counterexamples to the statement. (I like the letter X for eXceptions.) Since we have supposed the statement is false, $X \neq \emptyset$. By the Well-Ordering Principle, X contains a least element, x.

(Basis step.) We know that $x \neq 0$ because *show that the result holds for* 0*; this is usually easy.*

Consider $x - 1$. Since $x > 0$, we know that $x - 1 \in \mathbb{N}$ and the statement is true for $x - 1$ (because $x - 1 < x$). *From here we argue to a contradiction, often that* x *both is and is not a counterexample to the statement.* $\Rightarrow\Leftarrow$ ■

Proof Template 17 Proof by induction.

To prove every natural number has *some property*.

Proof.

- Let A be the set of natural numbers for which the result is true.
- Prove that $0 \in A$. This is called the *basis step*. It is usually easy.
- Prove that if $k \in A$, *then* $k + 1 \in A$. This is called the *inductive step*. To do this we
  - Assume that the result is true for $n = k$. This is called the *induction hypothesis*.
  - Use the induction hypothesis to prove the result is true for $n = k + 1$.
- We invoke Theorem 22.2 to conclude $A = \mathbb{N}$.
- Therefore the result is true for all natural numbers. ■

Proof Template 18 Proof by strong induction.

To prove every natural number has *some property*:

Proof.

- Let A be the set of natural numbers for which the result is true.
- Prove that $0 \in A$. This is called the *basis step*. It is usually easy.
- Prove that *if* $0, 1, 2, \ldots, k \in A$, *then* $k + 1 \in A$. This is called the *inductive step*. To do this we
  - Assume that the result is true for $n = 0, 1, 2, \ldots, k$. This is called the *strong induction hypothesis*.
  - Use the strong induction hypothesis to prove the result is true for $n = k + 1$.
- Invoke Theorem 22.9 to conclude $A = \mathbb{N}$.
- Therefore the result is true for all natural numbers. ■