CIT 5920
Recitation 11

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Penn Engineering

Overview for Today

- Logistics
- HW5 Most Popular Question Review
- Question 1 (3 minutes)
- Question 2 (3 minutes)
- Question 3 (4 minutes)
- Question 4 (5 minutes)
- Question 5 (6 minutes)

Logistics

- HW5 due 11:59 PM on Monday, November 25
  - Two duplicate questions have been removed. The updated homework file is now available on the course website. The assignment should include 7 exercises, with the first one titled "Contrapositive…"
- Midterm 2 has been canceled
  - The professor will adjust the weight of the remaining assessments upon returning
  - The final exam will still take place as scheduled
- No recitation on Wednesday, November 27 (Friday class schedule)

HW5 Most Popular Question Review

Vote for the question in HW5 that you are most interested in going over and we'll review it here!

Strong Induction Review

A technique for proving that a statement is true for all values of a variable:

Base case: prove that the statement is true for the first values of the variable

Inductive step: prove that if the statement is true for all values up to a certain point, it must also be true for the next value.

Weak vs Strong Induction Review

Weak induction: assumes that a statement is true at a specific step k

Strong induction: assumes a statement is true at all steps from the base case to the step k.

Useful when the result for n = k-1 depends on the result of some smaller value of n, but not necessarily the value that immediately precedes (k)

In other words: we use strong induction when the truth of the next step relies on the truth of multiple previous steps, not just the one immediately preceding.

Question 1

Exercise 1

For all integers,

Take the following statement: 'If a number leaves a remainder of 3 when divided by 4, then its square leaves a remainder of 1 when divided by 4.'

A. Rewrite the statement using logical symbols and implications.

B. Formulate the negation of the statement using logical symbols and implications.

Answer 1

Exercise 1

Take the following statement: 'If a number leaves a remainder of 3 when divided by 4, then its square leaves a remainder of 1 when divided by 4.'

A. Rewrite the statement using logical symbols and implications.

Solution: We can use mod to represent the remainder left when dividing by 4.
$\forall x, \left(x \equiv 3 \pmod 4 \rightarrow x^2 \equiv 1 \pmod 4\right)$

Answer 1 (cont.)

Exercise 1

Take the following statement: 'If a number leaves a remainder of 3 when divided by 4, then its square leaves a remainder of 1 when divided by 4.'

B. Formulate the negation of the statement using logical symbols and implications.

Solution: The original statement: $\forall x, (x \equiv 3 \mod 4 \rightarrow x^2 \equiv 1 \mod 4)$

Replace the Implication: $\forall x, (\neg (x \equiv 3 \mod 4) \vee (x^2 \equiv 1 \mod 4))$

Add Negation and Flip to Existential Quantifier: $\exists x, \neg(\neg(x \equiv 3 \mod 4) \vee (x^2 \equiv 1 \mod 4))$

Apply DeMorgan's law: $\exists x, \neg(\neg((x \equiv 3 \mod 4) \wedge \neg(x^2 \equiv 1 \mod 4)))$

Final Outcome: $\exists x, (x \equiv 3 \mod 4) \wedge (x^2 \not\equiv 1 \mod 4)$

Question 2

Exercise 2

Suppose you have 10 distinct integers. Show that there are two of these integers such that their difference is divisible by 9.

Answer 2

Exercise 2

Suppose you have 10 distinct integers. Show that there are two of these integers such that their difference is divisible by 9.

Solution: When dealing with 10 distinct integers, their possible remainders when divided by 9 are: $0, 1, 2, 3, 4, 5, 6, 7,$ and 8.

If we take the integers modulo 9, there are only 9 possible remainders.

According to the Pigeonhole Principle, since there are 10 integers and only 9 possible remainders when divided by 9, at least two of these integers must have the same remainder when divided by 9.

Let's say these two integers are a and b, where $a > b$.

The difference between these integers, $a - b$, would be divisible by 9, as the difference represents a multiple of 9 due to the equality of their remainders.

Detour: Fibonacci Numbers

- The first two numbers in the Fibonacci sequence are ones, which means:

$$F_1 = 1, \text{ and } F_2 = 1$$

- Every subsequent Fibonacci number is the sum of the two previous numbers in the sequence, which means:

$$F_n = F_{n-1} + F_{n-2}$$

- Altogether, the sequence looks like this:
  1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377…

Question 3

Exercise 3

Let F_i represent the ith Fibonacci number. Show that

$$\sum_{i=1}^{n} F_i = F_{n+2} - 1$$

Answer 3

Exercise 3

Let F_i represent the ith Fibonacci number. Show that

$$\sum_{i=1}^{n} F_i = F_{n+2} - 1$$

Solution: We know that $F_1 = 1$, $F_2 = 1$ and that $F_n = F_{n-1} + F_{n-2}$.

Base case

$F_1 = 1$ and we see $F_3 = 2$ and 2 -1 = 1 so it holds for 1.

Let us assume that

$$\sum_{i=0}^{k} F_i = F_{k+2} - 1$$

Answer 3 (cont.)

$$\sum_{i=0}^{k} F_i = F_{k+2} - 1$$

Now consider the summation

$$\sum_{i=0}^{k+1} F_i$$

This is basically the summation in the induction hypothesis with one extra term.

$$\begin{aligned}
\sum_{i=0}^{k+1} F_i &= \sum_{i=0}^{k} F_i + F_{k+1} \\
&= F_{k+2} - 1 + F_{k+1} \text{ by induction hyp.} \\
&= F_{k+3} - 1 \text{ since each fibonacci term is the summation of the two previous ones}
\end{aligned}$$

and this is what we wanted to show!

Question 4

Exercise 4

Use strong induction to prove that you can make any dollar amount greater than or equal to 4 dollars (don't worry about cents) using only 2 dollar and 5 dollar notes.

Answer 4

Exercise 4

Use strong induction to prove that you can make any dollar amount greater than or equal to 4 dollars (don't worry about cents) using only 2 dollar and 5 dollar notes.

Solution: Base cases:
We can see that we are going to need two base cases here.
Note that 4 can be made with two 2 dollar bills. 5 can be made with a single 5 dollar bill.

Inductive Hypothesis:
We will assume that all values from 4 to k can be made with 2 dollar bills and 5 dollar bills.

Inductive Step:
Now consider k+1. If k+1 happens to be 5, we have already shown how to make that amount.
So we will consider $k + 1 \geq 6$. Then $k + 1 - 2 \geq 4$. It falls within the region of the induction hypothesis. So we know how to make $k - 1$ with 2 and 5 dollar bills by the induction hypothesis. We can just add another 2 dollar bill and be done.
This coupled with the induction hypothesis will complete the proof by induction.

Question 5

Exercise 5

Prove using strong induction that every positive integer n has a binary expression.

Namely, we have to show that there exists $c_i \in \{0, 1\}$ such that:

$$n = c_r \cdot 2^r + c_{r-1} \cdot 2^{r-1} + \ldots + c_2 \cdot 2^2 + c_1 \cdot 2^1 + c_0$$

Answer 5

Exercise 5

Prove using strong induction that every positive integer n has a binary expression.

Namely, we have to show that there exists $c_i \in \{0, 1\}$ such that:

$$n = c_r \cdot 2^r + c_{r-1} \cdot 2^{r-1} + \ldots + c_2 \cdot 2^2 + c_1 \cdot 2^1 + c_0$$

Solution: Base cases $1 = 2^0$. Also $2 = 2^1$.

The induction hypothesis is the assumption that we know how to express any j, $1 \leq j \leq k$ using powers of 2.

Now let us consider $k + 1$. We need to show how to express this as powers of 2.

First thing to note is that $k + 1$ is either odd or even.

Case 1

If $k + 1$ is odd, then first using the induction hypothesis to express k in a binary expression.

Let us say $k = c_p 2^p + \ldots + c_1 2^1 + c_0$,

k is even. k is even would mean k is divisible by 2.

That would mean that the very last bit c_0 must be 0 (else it would leave a remainder of 1 when divided by 2).

That means that expressing $k + 1$ is easy. $k + 1 = c_p 2^p + \ldots + c_1 2^1 + 1$

Answer 5 (cont.)

Case 2

If $k + 1$ is even, then $k + 1 = 2t$ for some $t \in \mathbb{Z}$.

But this would mean that $t \leq k$

Therefore t actually is something that we can apply the induction hypothesis on.

$t = d_y 2^y + d_{y-1} 2^{y-1} + \ldots + d_1 2^1 + d_0$.

To get $k + 1$ we can get just multiply this whole expression by 2.

$k + 1 = d_y 2^{y+1} + d_{y-1} 2^y + \ldots + d_0 2^1$

This shows how to express $k + 1$ in binary in both the even and odd cases. That means that we have shown how to express $k + 1$ in binary in all cases.

Combine this with the two base cases and we have a complete proof by induction.

See you after Thanksgiving!

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