CIT 5920
Recitation 10

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Penn Engineering

Overview for Today

- Logistics
- Question 1
- Question 2
- Question 3
- Question 4
- Question 5

Logistics

- Homework 5 will be released today and will be due next Monday (Nov. 25).

Proof by Induction

To prove a statement by induction, we aim to show that it holds for an initial case, called the **base case**. Next, we assume it is true for some arbitrary integer k (the **induction hypothesis**) and then demonstrate that this implies it is also true for k + 1 (the **inductive step**).

This process establishes a chain: if the statement is true for one case, it logically follows for each subsequent case, thereby proving it for all cases in the specified range.

Components of Mathematical Induction

1. Define the mathematical claim as a predicate P(n). The statement should hold for all values n within a specified range, often $n \geq k$ for a starting integer k.
2. **Base Case:** Verify that P(k) is true for the smallest integer in the range, typically $k = 0$ or $k = 1$. This initial proof provides a foundation, confirming that the statement holds for at least one instance.
3. **Induction Hypothesis:** Assume P(k) is true for some arbitrary integer k. This assumption serves as a conditional step to support the proof of the next case, P(k+1).
4. **Inductive Step:** Using the induction hypothesis that P(k) is true, prove that P(k+1) must also be true (Show $P(k) \rightarrow P(k+1)$). This establishes that if the statement holds for k, it logically holds for $k + 1$ as well, completing the induction.

Induction Example

Prove that the sum of the first n natural numbers is n(n+1)/2

1. State P(n) and we will prove P(n) is true for all n >= 1
2. **Base Case:** when n = 1 we have 1 x 2 / 2 = 1. So the base case holds.
3. **Induction Hypothesis:** Assume P(k) is true for an arbitrary integer k, which is $\Sigma_{i=1}^{k} i = \frac{k(k+1)}{2}$
4. **Inductive Step:** Prove P(k+1) is true

Induction Example

Consider the sum of the first k + 1 natural numbers: $\Sigma_{i=1}^{k+1} i = \Sigma_{i=1}^{k} i + (k+1)$

By the induction hypothesis, we know that $\Sigma_{i=1}^{k} i = \dfrac{k(k+1)}{2}$

Substituting this into the expression gives: $\Sigma_{i=1}^{k+1} i = \dfrac{k(k+1)}{2} + (k+1)$

$$= \frac{k(k+1) + 2(k+1)}{2} = \frac{k^2 + k + 2k + 2}{2} = \frac{k^2 + 3k + 2}{2} = \frac{(k+1)(k+2)}{2}$$

Therefore, $\Sigma_{i=1}^{k+1} i = \dfrac{(k+1)((k+1)+1)}{2}$ which matches the formula for n = k + 1.

Question 1

Prove the statement: An irrational number multiplied by a nonzero rational number will always result in an irrational number.

Hint: Proof by contradiction

Answer 1

Solution: Suppose $a \in \mathbb{Q}$ and $b \in \overline{\mathbb{Q}}$ (and $a \neq 0$).
Now assume for contradiction that their product $ab \in \mathbb{Q}$, so then $\exists p, q \in \mathbb{Z}$ such that $q \neq 0 \wedge ab = \frac{p}{q}$.
We can divide both sides by a (since it's nonzero) to get $b = \frac{p}{qa}$. Now since a is rational, $a = \frac{c}{d}$ for $c, d \in \mathbb{Z}$ and $d \neq 0$.
Thus $b = \frac{pd}{cq}$. However, since c and q are nonzero, cq is nonzero, and since pd and cq are integers, b must be rational.
This is a contradiction, so it must be that $ab \notin \mathbb{Q}$.

Question 2

Show that $4^{n-1} > n^2$ for $n \geq 3$ by mathematical induction.

Answer 2

Solution:
Base Case: When $n = 3$, we have $4^2 = 16 > 9$, so the statement holds.
Induction Hypothesis: Assume the statement holds when $n = k$, so $4^{k-1} > k^2$ for some $k \geq 3$.
Induction Step: We now want to show that it holds for $n = k + 1$. WWTS that $4^k > (k + 1)^2$
First of all, by IH, we know that $4^{k-1} > k^2$. Multiplying both sides by 4 gives $4^k > 4k^2$.
Now our goal is to somehow show that $4k^2$ *will be larger than* $(k + 1)^2$
To do this, consider some variable y with following value:

$$y = 4k^2 - (k+1)^2$$
$$y = (2k + k + 1)(2k - k - 1)$$
$$y = (3k + 1)(k - 1)$$

Since $k \geq 3$, $k - 1$ is greater than 0, and $3k + 1$ is greater than 0, and we know by multiplying two positive numbers will give a positive number as well. Therefore y is greater than 0, which means:

$$4k^2 > (k+1)^2$$

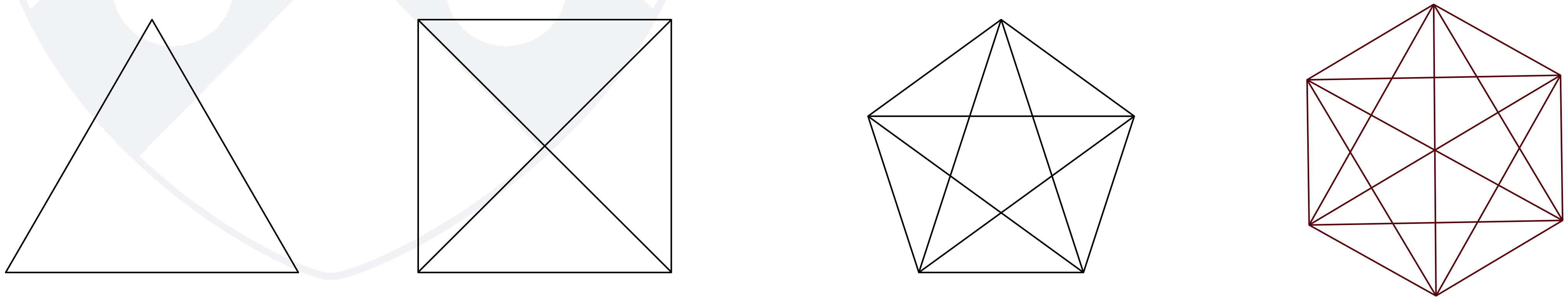
Therefore:

$$4^k > 4k^2 > (k+1)^2$$

so we have successfully shown what we wanted to show, that $4^k > (k + 1)^2$.

Question 3

We saw in an earlier recitation that a n-sided polygon (convex) has $\dfrac{n(n-3)}{2}$ diagonals. We did this with a combinatorics argument. Now prove this using induction!



Answer 3

Solution:

Base Case (BC): When $n = 3$, we have $\frac{3(3-3)}{2} = 0$, and there are indeed 0 diagonals in a triangle.

Induction Hypothesis (IH): Suppose $n = k$ holds. That is, for a k-sided (convex) polygon, there are $\frac{k(k-3)}{2}$ diagonals.

Induction Step (IS): Now we want to show that for a $(k+1)$-sided (convex) polygon, the number of diagonals is $\frac{(k+1)(k-2)}{2}$.

Consider a (convex) polygon with $k+1$ sides. Pick any vertex arbitrarily at random and remove it (and join its previously adjacent edges). We are now left with a polygon with k sides, so by IH, there are $\frac{k(k-3)}{2}$ diagonals in this polygon. Now notice that these diagonals in this polygon still remain in the new polygon when we add that last vertex back in. However, we also gain $k-2$ diagonals since we need to connect this last vertex to the other $k-2$ non-adjacent vertices.

Furthermore, we gain one more diagonal between the two adjacent vertices that used to be an edge. Thus the number of diagonals in this $(k+1)$-sided polygon is $\frac{k(k-3)}{2} + (k-2) + 1 = \frac{k^2-3k}{2} + \frac{2k-2}{2} = \frac{k^2-k-2}{2} = \frac{(k+1)(k-2)}{2}$. This is exactly what we wanted to show.

Question 4

Use induction to prove the following theorem

Theorem. For any integer $n \geq 2$, in a group of n friends where everyone shakes everyone else's hand exactly once, $\frac{n(n-1)}{2}$ handshakes take place.

Answer 4

Let $P(n)$ be the statement that in a group of n friends where everyone shakes everyone else's hand exactly once, $\frac{n(n-1)}{2}$ handshakes take place.

Base case: We verify that $P(2)$ is true. In a group of 2 friends, $\frac{2(2-1)}{2} = 1$ handshake takes place. Thus, $P(2)$ is true.

Induction hypothesis: Assume $P(k)$ is true for some $k \geq 2$.

Inductive step: We will show that $P(k+1)$ is true.
By the induction hypothesis, we know that in a group of k friends, $\frac{k(k-1)}{2}$ handshakes take place. The additional person shakes hands with each of the k people in the original group, adding k more handshakes. So the total number of handshakes in a group of $k+1$ people is:

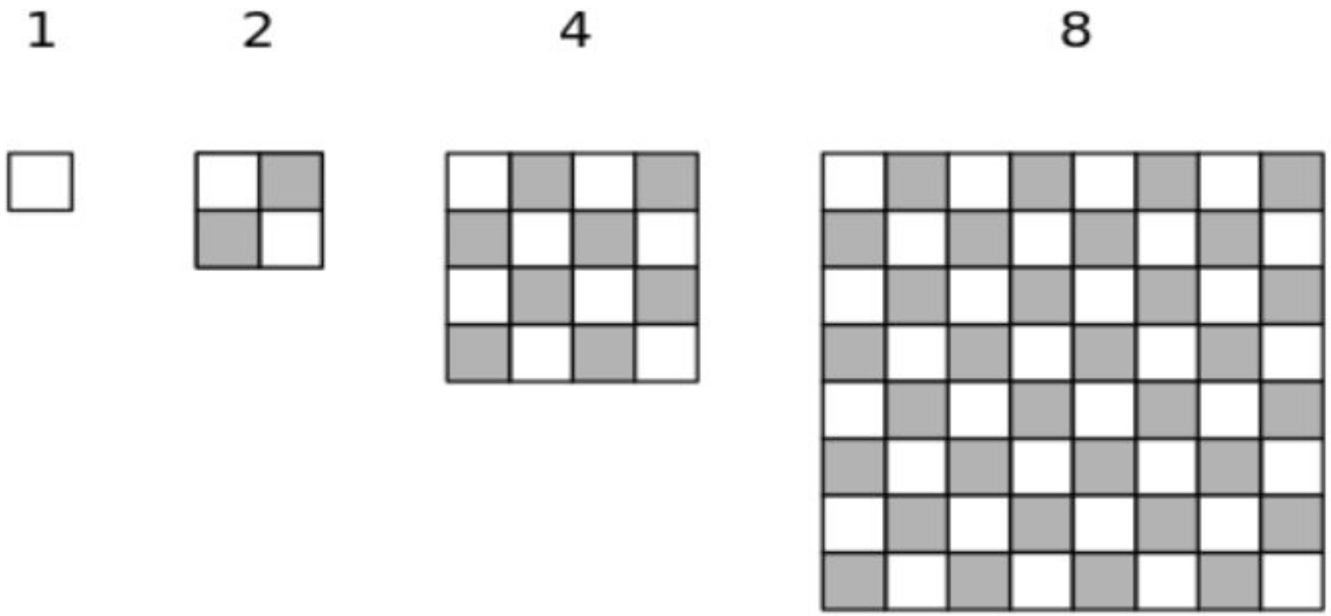
$$\frac{k(k-1)}{2} + k = \frac{k(k-1)+2k}{2} = \frac{k(k+1)}{2} = \frac{(k+1)((k+1)-1)}{2}$$

Therefore, $P(k+1)$ is true.

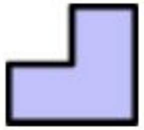
By induction, $P(n)$ is true for all $n \geq 2$.

Question 4

Consider a checkerboard grid like one of the following:



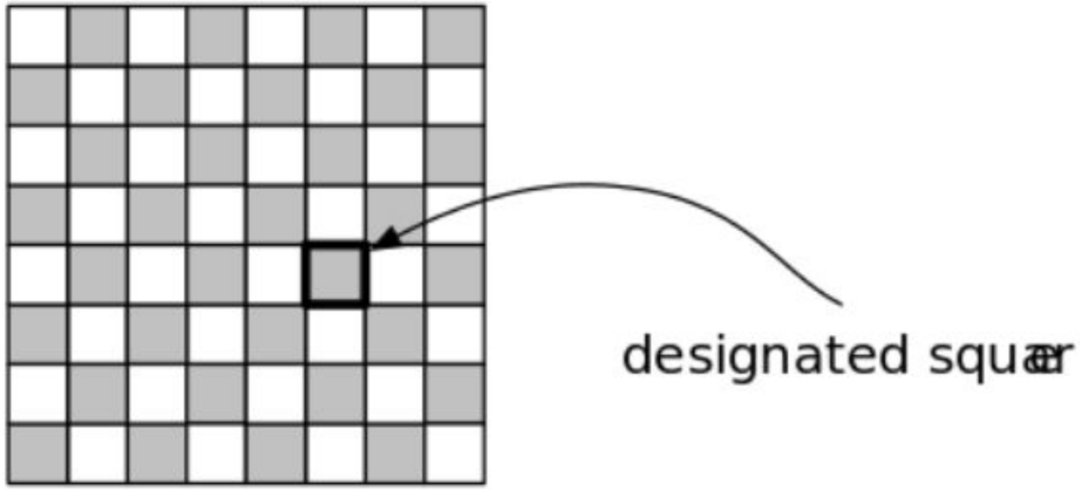
Also consider the following "corner piece" that can go in any orientation/rotation.



https://nstarr.people.amherst.edu/trom/puzzle-8by8/

Question 4

Prove by induction that for all integers $n \geq 1$, we can tile a 2^n by 2^n checkerboard with these pieces, leaving a single, arbitrary, square uncovered.



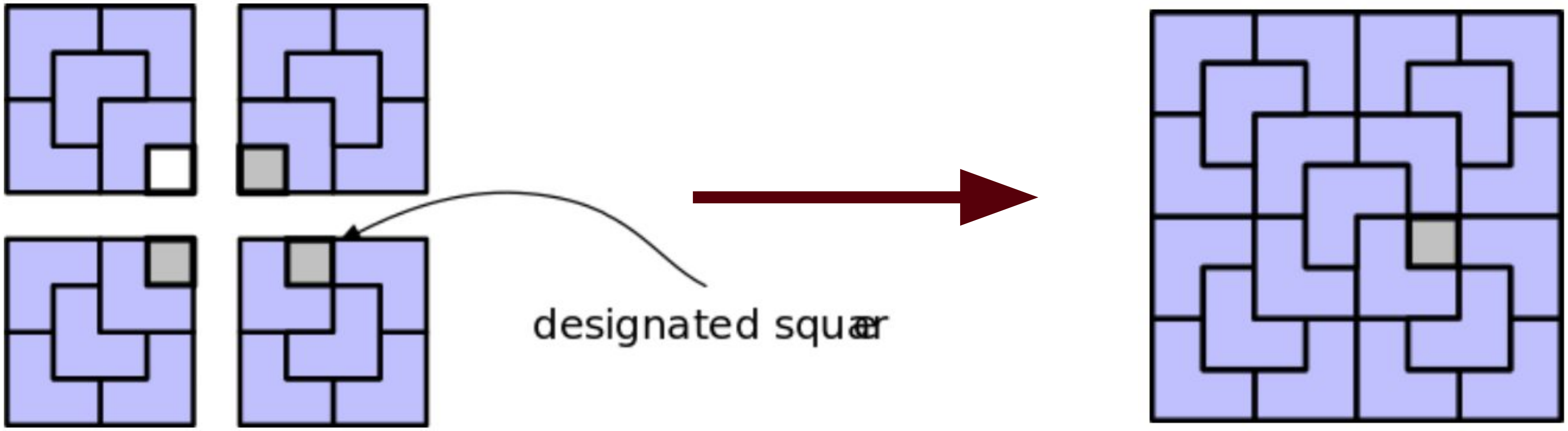
For a hint, remember, we are trying to use induction. Can you try to start from a small case and build your way up to a bigger case?

Answer 4

Solution:
Base Case: Consider a 2 by 2 grid. Clearly no matter which arbitrary square is chosen to be empty, we can rotate our corner piece to leave that one empty, but tile/fill every other square
Induction Hypothesis: Now assume that for some integer $k \geq 1$, for a board of size 2^k by 2^k, we can tile it such that we can leave any single, arbitrary tile empty while everything else is covered.
Induction Step: Now we want to show that for a board of size 2^{k+1} by 2^{k+1}, we can again tile it leaving any one square empty. First of all, WLOG let this empty square be in the bottom right "quadrant" (which exists since the size is even), and by symmetry we can just rotate everything to achieve this. Then in this 2^k by 2^k size quadrant, by IH we can tile it such that this arbitrary location is empty, but every other location in the quadrant is filled. Now for the other 3 quadrants, again by IH, we can tile them such that they are all filled except the three "central" most tiles are left uncovered. That is, the tile closest to the center of the large square in each of the other 3 quadrants is left uncovered (we can do this again since we can leave any arbitrary tile empty). See the image below.

Answer 4 (cont.)

Then we can use one more "corner piece" to cover these three pieces no matter what rotation they are in. Now we have successfully accomplished the goal of tiling the square such that only that single, arbitrary tile is left uncovered.



Question 5

Use induction to show that $6^n + 4$ is divisible by 5 for every $n \geq 1$ and $n \in \mathbb{Z}$.

Answer 5

Solution:
Base Case: When $n = 1$, we have $6^1 + 4 = 10$, and 10 is indeed divisible by 5 (by definition).
Induction Hypothesis: Assume the statement holds when $n = k$, such that $6^k + 4$ is divisible by 5 for some $k \in \mathbb{Z}$
Induction Step: Now we want to prove it holds for $n = k + 1$, that is, that $6^{k+1} + 4$ is divisible by 5.
We have

$$\begin{aligned} 6^{k+1} + 4 &= 6 \cdot 6^k + 4 \\ &= (5 + 1) \cdot 6^k + 4 \\ &= 5 \cdot 6^k + 6^k + 4 \end{aligned}$$

By IH, we know $6^k + 4$ is divisible by 5, and we also know $5 \cdot 6^k$ is divisible by 5. Therefore, we must have that $6^{k+1} + 4$ is divisible by 5.

Therefore, we have shown by induction that $6^n + 4$ is divisible by 5 for all integers $n \geqslant 1$.

See you next week!

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