Proof Selection

* Direct Proof: to prove $A \Rightarrow B$ when it is straightforward to go from A to B, and A and B are simple to define.

* Proof by contrapositive: to prove $\underline{A} \Rightarrow \underline{B}$, where either (i) A is "complicated" or B is "complicated" but not B is "simple".

* Proof by Contradiction: is used to prove "existence" ("does not exist") OR "uniqueness" ("is not unique").

* Proof by Induction: any statement that can be indexed by integers, and is true for all integers after a certain point.

Ex. 1. Prove that $2\pi + 3$ is irrational. You can assume $\underline{\pi \text{ is irrational}}$.

Suppose $2\pi + 3$ is $\underline{\text{rational number}}$.

$$2\pi + 3 = \frac{a}{b} \text{ for some } \underline{a, b \in \mathbb{Z}}, \ b \neq 0$$

Then $2\pi = \frac{a}{b} - 3 = \frac{a-3b}{b}$

$$\pi = \frac{\frac{a}{b} - 3}{2} = \frac{a-3b}{b} \cdot \frac{1}{2} = \frac{a-3b}{2b}$$

Since $a - 3b$ is an integer, $2b$ is integer.
hence $\frac{a-3b}{2b}$ is a rational number,
$\therefore \underline{\pi \in \mathbb{Q}}$.

We have a contradiction.

Ex.2.

A B

Prove that for all integers x, y. if $(x^2+1)(y+1)$ is even. then x is odd or y is odd.

$\neg B \rightarrow \neg A$

Suppose that neither x or y is odd, which is both x and y is even

$x^2 = x \cdot x$ is even since a product of evens is even.

x^2+1 is odd since even + odd = odd.

$y+1$ is odd since even + odd = odd

$(x^2+1)(y+1)$ is odd since product of odds is odd.

Thus $(x^2+1)(y+1)$ is not even.

Ex. 3. If $d|a$ and $d|b$, then $d|(a-2b)$

Suppose $a = dn$, $n \in \mathbb{Z}$,

$b = dm$, $m \in \mathbb{Z}$,

$a-2b = dn - 2(dm)$

$= dn - d(2m)$

$= d(n-2m)$

$n \in \mathbb{Z}$, $2m \in \mathbb{Z}$, $\Rightarrow n-2m \in \mathbb{Z}$.

$d | a-2b$ is true. $\square$

4. c) $\sum_{i=1}^{n} i^2 = n(n+1)(2n+1)/6$

BC smallest possible n ?? $n=1$

Left side (LHS) $\sum_{i=1}^{1} 1^2 = 1$

RHS $\frac{1 \cdot 2 \cdot 3}{6} = 1$

$\therefore$ BC done

$1^2+2^2+\ldots(k-1)^2+k^2$

IH assume $\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$ ①

IS We want to show (WTS)

$$\sum_{i=1}^{k+1} i^2 = \frac{(k+1)(k+2)(2(k+1)+1)}{6}$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

$$\sum_{i=1}^{k+1} i^2 = \boxed{\sum_{i=1}^{k} i^2} + (k+1)^2 \quad ②$$

By IH

$$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

$$= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6}$$

$$= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6}$$

$$= (k+1)\left[\frac{2k^2+k+6k+6}{6}\right]$$

$$= (k+1)(2k+3)(k+2)/6$$

$$(k+2)(2k+3) = 2k^2 + 4k + 3k + 6$$

Q5 $\sqrt{2}$ + rational number is irrational.

(a) direct

$p \Rightarrow q$

add $\sqrt{2}$ + rational

result of sum is irration

"if rational number is added to $\sqrt{2}$ then result is irrational"

start - add any ~~irr~~ rational number to $\sqrt{2}$

end - to show sum is irrational.

note: for this Q we'll assume $\sqrt{2}$ is irrational is proven.

(b) if the ~~sum of~~ result of the addition is ~~not irrational~~ rational

then we did not add rational number to $\sqrt{2}$.

~~attempt~~ assume $\frac{p}{q} = \sqrt{2} + ?$ not rational ??

(c) Contradiction

Assume $\sqrt{2}$ + rational number

and result is rational.

(1)

$$\sqrt{2} + \frac{p}{q} = \frac{m}{n}$$

$$\therefore \quad \sqrt{2} = \frac{m}{n} - \frac{p}{q}$$

$$= \frac{mq - np}{nq}$$

is rational

But $\sqrt{2}$ is not rational !!

CONTRADICTION