Practice Problems: Mathematical Proofs

Proof by Contradiction, Contrapositive, Direct Proof, and Induction

Instructions: For each of the following statements, write a proof using the indicated proof technique.

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1. Proof by Contradiction:
Prove that $2\pi + 3$ is irrational. You may assume that π is irrational.

2. Proof by Contrapositive:
Prove that for all integers x, y, if $(x^2 + 1)(y + 1)$ is even, then x is odd or y is odd.

3. Direct Proof:
Prove that for all integers a, b, c, if $d|a$ and $d|b$, then $d|(a - 2b)$

4. Proof by Induction:
Prove that for all integers $n \geq 1$, show:

(a)
$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

(b)
$$\sum_{i=1}^{n} 2i - 1 = n^2$$

(c)
$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

(d)
$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

5. Proof Selection:
Consider the statement: The sum of $\sqrt{2}$ and a rational number is irrational. For a,b,c, Show that for each method, how you can start and end.

(a) How should a direct proof of this statement start and end? That is, what do you assume, what do you try to prove?

(b) How should an indirect proof by contraposition start and end?

(c) How should a proof by contradiction start? What should you be trying to show (in general terms)?

(d) Give a correct proof of the statement.

6. Strong Induction:
In the parlour game Nim, there are two players and two piles of matches. At each turn, a player removes some (non-zero) number of matches from one of the piles. The player who removes the last match wins

If the two piles contain the same number of matches at the start of the game, then the second player can always win.

Hints:

- For Proof by Contradiction: Assume the negation of the statement you want to prove, and derive a contradiction.
- For Proof by Contrapositive: Rewrite the statement in contrapositive form and prove that instead.
- For Direct Proof: Use definitions and properties directly to establish the result.
- For Proof by Induction: Prove the base case, and then prove the inductive step.
- even and odd properties:

  1. $$EVEN * EVEN = EVEN$$

  2. $$EVEN + ODD = ODD$$

  3. $$ODD * ODD = ODD$$