CIT 5920 – Induction

review of weak induction

intro to strong induction

examples / practice.

$P(k) \Rightarrow P(k+1)$

Q. By induction prove that $5^n + 3$ is divisible by 4 $\forall n \in \mathbb{Z}^+$ $n = \{1, 2, 3, \dots\}$

Proof by induction

1. Base case – show holds for smallest val
2. Induction hypothesis (IH) – statement holds for some $n = k$ (assume)
3. Induction step – using IH show statement holds for $n = k+1$.

Base Case for $n = 1$

$5^1 + 3 = 8$. 8 is $2 * 4$. Clearly divisible by 4. done

IH assume $5^k + 3$ is divisible by 4 for some $k \in \mathbb{Z}^+$

Ind step We need to show $5^{k+1} + 3$ is divisible by 4

Goal: Somehow bring in IH

$$5^{k+1} + 3 = 5^k \cdot 5 + 3 \quad (1)$$

$$= \boxed{5(5^k + 3)} + \underbrace{3 - 15}_{-12} \quad (2)$$

by IH we know $5^k + 3$ is div by 4 $\therefore$ $5(5^k+3)$ also div by 4

clearly div by 4 $-12 = 4 * (-3)$

$\therefore$ $5^{k+1} + 3$ is div by 4

which is what we needed to show
done by induction.

$n = a$

STRONG INDUCTION

① Base cases: Show statement holds for some small values.

② Ind hypothesis (IH): assume statement holds for all values from the base case upto and including k.

$P(a) \wedge P(a+1) \wedge \dots \wedge P(k)$ assumed to be true

③ Ind step
using IH show $P(k+1)$ i.e. statement holds for $k+1$.

Example 1 Prove that all positive integers ≥ 2 are either prime or can be expressed as a product of primes.

define prime — only factors are 1 and itself.

Is 1 prime ?? N

$\underline{n}$

all numbers ≥ 2 are either prime or composite

composite → not prime

has more than 2 factors

$\exists$ some factor other 1 and itself.

$\exists a \quad 1 < a < n, \ a \in \mathbb{Z}^+$
such that $a \mid n$.

$a \mid b$
a divides b
OR
$b = ak$.

Prove that all positive integers ≥ 2 are either prime or can be expressed as a product of primes.

Base case $n = 2$ 2 is prime. done.

IH Assume result holds $\forall m \in \mathbb{Z}^+$ st

$$2 \leq m \leq k$$

Induction step to show result for $k+1$

$k+1$

Case 1: if $k+1$ prime done

Case 2: if $k+1$ is composite.

$\exists\, a$ st $a \mid k+1$ & $1 < a < k+1$

$\therefore\ 2 \leq a \leq k$

a is within range where IH is applicable.

applying IH to a gives us a is either a prime or a product of primes.

$$a \to p_1 \quad \text{OR} \quad a \to p_1 p_2 \cdots p_r$$

Since $a \mid k+1$ $\therefore$ $k+1 = a * b$

a: prime or product of primes

b: also $1 < b < k+1$ $\therefore$ we can apply IH on b also

applying IH to b

$$b \to q_1 \text{ where } q_1 \text{ is prime} \quad \text{OR} \quad b \to q_1 q_2 \cdots q_c \text{ (product of primes)}$$

By IH a is prime / product of primes
b is prime / product of primes

$k+1 = ab \rightarrow$ product of primes.

Which shows result.

$100 = 25 \times 4$
$\quad 5 \times 5 \quad 2 \times 2$

$99 = 33 \times 3$
$\quad 11 \times 3 \times 3$

Ex 2: n^{th} fibonacci $\leq 2^n$

$fib(1) = 1 \qquad fib(2) = 1$

$fib(n) = fib(n-1) + fib(n-2)$ Recursion

Base case $fib(1) = 1 \quad 2^1 = 2$
clearly $1 \leq 2$.

also $fib(2) = 1 \quad 2^2 = 4$
clearly $1 \leq 4$

Result holds for 1^{st} term & 2^{nd} term

IH for $fib(1), fib(2), \ldots, fib(k)$
$fib(i) \leq 2^i$

$\boxed{k+1 \geq 3}$

Ind step Want to show $fib(k+1) \leq 2^{k+1}$

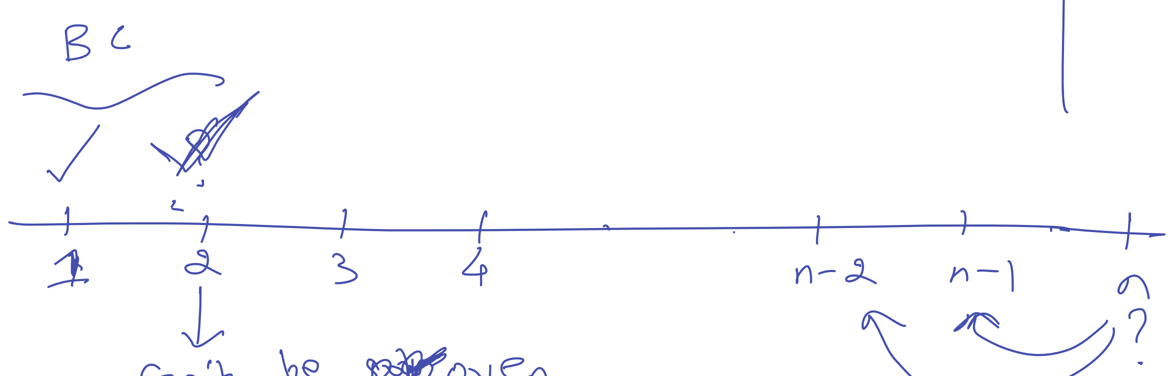
$\| $

$fib(k) + fib(k-1)$

$\leq 2^k \qquad \leq 2^{k-1}$

eventually
$\frac{1}{2}$ ✓

3
⋮
n



can't be ~~pp~~ proven by IH

$$\text{fib}(k+1) \le 2^{k+1}, \quad k+1 \ge 3$$

$$\parallel$$

$$\text{fib}(k) + \text{fib}(k-1)$$

by IH $\text{fib}(k) \le 2^k$ & $\text{fib}(k-1) \le 2^{k-1}$

$$\therefore \text{fib}(k+1) \le 2^k + 2^{k-1}$$

$$\le \underbrace{2^k + 2^k}_{2^{k+1}}$$

we know $2^{k-1} < 2^k$

done

Ex 3 Assume we have currency system where 3 cent coin, 4 cent coin and nothing else.

infinite supply

Q. For what values of n cents can we generate exact change??

$$\begin{aligned} 6 &= 3 * 2 \\ 7 &= 3 + 4 \\ 8 &= 4 * 2 \\ 9 &= 3 * 3 \\ 10 &= 3 + 3 + 4 \\ &\vdots \end{aligned}$$

Claim: $\forall\, n \in \mathbb{Z}^+,\ n \geq 6$ we can generate exact change.

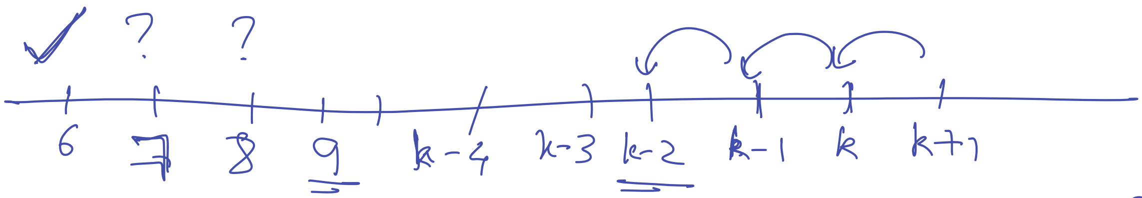
3 cent
4 cent

Proof by ind

B.C. clearly 6 is possible
we also need B.C. of 7 & 8

IH assume that $\forall\ 6 \leq i \leq k$ that i cents can be generated exactly.

Induction step: want to show $k+1$ cents can be generated exactly.



If we know by IH that $k-2$ cents is generated exactly.

if

$$3\begin{cases} k-2 = 3p + 4q \\ k+1 = 3(p+1) + 4q \end{cases}$$

(take one more 3 cent coin)