Induction

Recognizing a pattern

ex. $1+3 = 4 = 2^2$

$1+3+5 = 9 = 3^2$

$1+3+5+7 = 16 = 4^2$

odd numbers

$$\left(\begin{array}{c}\text{Sum of odd Numbers} \\ \text{up until } N\end{array}\right) = N^2$$

$$P(N): \sum_{i=1}^{N} (2i-1) = N^2$$

Idea behind Induction

Induction is used to prove a theorem holds for all non-negative integers $(N \geq 0)$ if we are able to do the following:

* State the proposition as depending on an integer parameter $P(N)$

* Show the base case, $P(k)$ is verified for $k=1$ (or k equals a small number), (the base case must exist but not always 0 or 1)

* Show that if theorem holds for k then it must also $k+1$

$k \Rightarrow k+1$, $k-1 \Rightarrow k$ the same thing

⇒ Assume theorem holds for k, ⇒ induction hypothesis

⇒ Prove theorem holds for k+1 by assumption ⇒ induction step.

When we have all three elements, we can prove that

$P(N)$ holds for all $N \geq k$ for a certain k.

Intution

* the induction step says that

if $P(k)$ then $P(k+1)$

that means

if $P(1)$ then $P(2)$ base case

but if $P(2)$ then $P(3)$

if $P(3)$ then $P(4)$ ↓

⋮

ex. Let $P(k)$ be the property

$$P(k) : \sum_{i=1}^{k} 2i-1 = k^2$$

Base Case: we have to identify which value of k makes sense to begin with. (usually $k=0$ / $k=1$)

Since it needs to be proved on all odd numbers, we choose $k=1$

For $k=1$:

LHS: $\sum_{i=1}^{1} 2i-1 = 2\times1-1 = 1$

RHS: $1^2 = 1$

$\because$ LHS = RHS, $\Rightarrow P(1)$ is true

Inductive step:

We assume $P(k)$ is true, we want to show $P(k+1)$ is true

$P(k)$ is true $\Rightarrow \sum_{i=1}^{k} 2i-1 = k^2$

$P(k+1)$: $\sum_{i=1}^{k+1} 2i-1 = (k+1)^2$

$$\sum_{i=1}^{k+1} 2i-1 = \sum_{i=1}^{k} 2i-1 + 2(k+1)-1$$

$$= k^2 + 2k+2-1$$

$$= k^2+2k+1$$

$$= (k+1)^2$$

$\Rightarrow P(k+1)$ is true.

Therefore, we show that $P(k) \Rightarrow P(k+1)$

Since base case + Inductive Step then we show

$P(n)$ is true for all $n \geq 1$, $\sum_{i=1}^{n} 2i-1 = n^2$ is true

Example with base case different than $k=0$ or $k=1$

Question: Is $N!$ bigger than 2^N?

Exploration Ⓐ Computing for small values of N

N	$N!$		2^N
1	1	$<$	2
2	2	$<$	4
3	6	$<$	8
4	24	$>$	16
5	120	$>$	32

Ⓑ Reasoning abstractly N terms for large values

$$
\begin{array}{lcccccccc}
N! & 1 \cdot & 2 & 3 \cdot & 4 \cdot & 5 \cdot & - & - & - \\
 & \wedge & \wedge & \wedge & \vee & \vee & \vee & \vee & \vee \\
2^N & \underbrace{2 \cdot & 2 \cdot & 2 \cdot & 2 \cdot & 2 & \cdots}_{N \text{ terms}}
\end{array}
$$

Things I've uncovered:

- $N!$ and 2^N are products of N terms
- all but the first three terms of $N!$ are larger than 2^N
- Therefore, we expect $N! > 2^N$, $\boxed{N \geq 4}$

We want to prove $P(N): N! > 2^N$ is true for all $N \geq 4$.

Base Case: let $N = 4$, then LHS : $N! = 24$, RHS : $2^N = 16$

Since $24 > 16$, we say $P(4)$ is true

Inductive Step: We assume that $P(k)$ is true which means $k! > 2^k$

We want to know whether this pattern continues with

$(k+1)!$ and 2^{k+1}

LHS RHS

$$(k+1)! = \underline{k!} \times (k+1)$$

Since $k! > 2^k$ from inductive hypothesis

$$(k+1)! > \underline{2^k} \times \underline{(k+1)} \qquad k+1 > 2$$

$$(k+1)! > 2^k \times \underline{2}$$

$$(k+1)! > 2^{k+1}$$

therefore $P(k) \Rightarrow P(k+1)$

By base case and Inductive step, we proved for all $N \geq 4$, $N! > 2^N$

Receipe to find a base case :

① try 1,

② if that doesn't work, try 2

③ if that doesn't work, investigate other options

Factorial Algorithm

Factorial is a function defined mathematically by

$$\underline{n!} = \begin{cases} 1 & \text{if } n=0 \\ n\times(n-1)! & \text{if } n>0 \end{cases}$$

In Java:

```
int FACT (int n) {
        if (n==0) Return 1;
        else Return n * FACT (n-1);
}
```

$P(n)$: $FACT(n) = n!$

Base Case: The base case is $n=0$

by definition of factorial, $0! = 1$ (RHS)

by definition of FACT, FACT(0), the function returns 1 because $n==0$ is true.

This shows that $FACT(0) = 0!$, base case is true. $P(0)$ is true.

Inductive Step: Let's assume our Property is true for n, $P(n)$ is true. we want to show $P(n+1)$ is true.

Inductive Hypothesis: $FACT(n) = n!$

By the algorithm, when you call $FACT(n+1)$, it will compute

$(n+1) * FACT((n+1)-1) \Rightarrow (n+1) * FACT(n)$

By the inductive hypothesis, we know that $FACT(n) = n!$

We can substitute:

$$(n+1) * FACT(n) = (n+1) * n!$$
$$= (n+1)!$$

So, we show that $P(n) \to P(n+1)$ and combined with the base case. $P(n)$ folds for all n. $n \geq 0$.

Practice:

Sum of first N integers: $\underline{\sum_{i=1}^{n} i} = \frac{n(n+1)}{2}$

$$P(n): \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Base Case: $n=1$. LHS: $\sum_{i=1}^{1} i = 1$, RHS: $\frac{1\times(1+1)}{2} = \frac{1\times 2}{2} = 1$

LHS = RHS, $\Rightarrow P(1)$ is true

Inductive Step: Assume $P(k)$ is true, we want to show $\underline{P(k+1)}$ is true.

Show: $\underline{\sum_{i=1}^{k} i = \frac{k(k+1)}{2}}$. $\Rightarrow \underline{\sum_{i=1}^{k+1} i} = \frac{(k+1)(k+1+1)}{2} = \frac{(k+1)(k+2)}{2}$.

$$\sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + k+1 = \frac{k(k+1)}{2} + k+1$$
$$= \frac{k(k+1)}{2} + \frac{2(k+1)}{2}$$
$$= \frac{(k+1)(k+2)}{2}$$

Therefore. $P(k) \Rightarrow P(k+1)$.

Based on base case and Inductive step, we proved

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \text{ is true.}$$