Prove Conversation

A: For every integer x. if x is even, then x^2 is even

B: Why is it true?

A: Give me any integer x. and I'll show you why this is true

B: What about $x=17$

A: 17 is not an even number, it is vacuously true.

B: $x=62$,

A: $62^2 = 3844$. even number

B: We show $x=17$, $x=62$ is true. but how do you know it is true for every integer

A: Let x be any integer, if x is even, x^2 is even

By the definition of even, there exists an integer y s.t. $x=2y$, square both sides. $x^2 = 4y^2 = 2 \cdot 2y^2$

Done.

Ex For every integer x, if x is odd, then $x+1$ is even

$$\forall x \in \mathbb{Z}.\ Odd(x) \rightarrow Even(x+1)$$

$(\forall x \in S) P(x)$: let x be any element of S, prove $P(x)$ is true

predicates

Let x be any integer, Suppose x is odd,

there exists an integer y such that $x = \underline{2y+1}$

$x+1 = 2y+1+1 = 2y+2 = 2(y+1)$, Assume $k = y+1$,

$x+1 = 2k$, $k \in \mathbb{Z}$,

x is even.

QED $\square$

Logic Review:

Propositions: Complete Statements that are definitely true or false

Predicates: Statements involving variables that become proposition

Boolean Variables: conjunction '$\wedge$', disjunction '$\vee$', NOT '$\neg$'

Implications: $P \to q$, if P then q, Q if P, P only if Q

Quantifiers: Universal, $\forall$, for all

Existential, $\exists$, exists

Contrapositive, Converse and Inverse

Contrapositive: $\neg q \Rightarrow \neg p$, $p \Rightarrow q$

Converse : $q \Rightarrow p$

Inverse : $\neg p \Rightarrow \neg q$

Indirect Proof

Proof by contradiction

Proof by contrapositive

Contradiction:

Assume p is false, find the contradiction ($\neg P$)

Ex. Today is a rainy day $\Rightarrow P$

Assume it is not a rainy day, then we found the ground is wet, Contradiction! $\Rightarrow \neg P$ is false.

Today is a rainy day. $\Rightarrow P$ is true

Assumption opposite

$P \Rightarrow q$, $\quad \neg(P \Rightarrow q) \Rightarrow$ false

$\neg(\neg P \vee q) \Rightarrow$ false

$P \wedge \neg q \Rightarrow$ false

Pigeonhole Principle

Statement. If [N pigeons are placed into M pigeonholes with N > M] (A), then [at least one pigeonhole must contain more than one pigeon] (B)

$A \Rightarrow B$, $\quad [A \wedge \neg B \Rightarrow \text{false}]$

① Assume the opposite:

A

Suppose N pigeons are placed into M pigeonholes with $N > M$,

and every pigeonhole contains at most one pigeon.

$\neg B$

② This means that maximum number of pigeons that can be placed without any pigeonhole containing more than one pigeon is M.

③ Build up to contradiction:

But we have N pigeons and $N > M$, this contradicts our original assumption that every pigeonhole contains at most one pigeon.

④ Conclude that $\neg B$ can't be true

Therefore, $\neg B$ is false and the pigeonhole principle is true.

Proof by Contrapositive $p \Rightarrow q$, $\boxed{\neg q \rightarrow \neg p}$

if $\underbrace{\text{[No pigeonhole contains more than one pigeon]}}_{\neg B}$, then $\underbrace{N \leq M}_{\neg A}$

① Assume no pigeonhole contains more than one pigeon ($\neg B$).

② This means the maximum number of pigeons that can be placed without any pigeonhole containing more than one pigeon is M.

③ Therefore, the number of pigeons (N) must be less than or equal to M.

④ This proves the contrapositive ($\neg B \rightarrow \neg A$) $\square$

Ex. Prove $\sqrt{2}$ is not rational.

Proof by contradiction:

Rational:

Def: A Number x is rational if and only if there exists $p, q \in \mathbb{Z}$, s.t. $x = \frac{p}{q}$, $q \neq 0$

p, q are coprime

① Assume Contradiction

Let's assume $\sqrt{2}$ is rational,

② $\exists\ p, q \in \mathbb{Z}$, with $q \neq 0$, s.t. $\sqrt{2} = \frac{p}{q}$

③ Derive until we find contradiction.

④ $\sqrt{2} = \frac{p}{q}$ (Square by both sides)

$$2 = \frac{p^2}{q^2}$$

$2q^2 = p^2$, p^2 is even number (Since divisible by 2)

$\rightarrow$ p is even number ①

⑤ Therefore there exists $k \in \mathbb{Z}$, s.t. $p = 2k$,

$$2q^2 = (2k)^2$$

$$q^2 = \frac{4k^2}{2}$$

$$q^2 = 2k^2$$

$\Rightarrow q^2$ is even number $\Rightarrow$ q is even number ②

From ① + ②, if both p and q are even then they share a common factor, they are not coprime, which conflicts with our original assumption that p and q are coprime

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⑥ Therefore our original assumption that $\sqrt{2}$ is rational must be false

⑦ $\sqrt{2}$ is irrational

Q.E.D.