# CIT 5920–Mathematical Foundations of Computer Science
# Homework 6: *Introduction to Proofs*

Version: November 23, 2024

Complete the online portion on PrairieLearn. For the written section, use the **customized HW6 template on Overleaf**, shared on the course forum and Canvas. Each exercise should be on a separate page. Only submissions in this format will be accepted. Submit your written work on Gradescope by the deadline. For assistance or inquiries, don't hesitate to: Attend office hours; post questions on the class forum; ask about the motivation behind this material.

Guidelines:

- All guidelines from previous homeworks apply.
- In particular, write explanations unless you are explicitly told that we do not require them.

## Exercise 0 – PrairieLearn Questions [ 50pts ]

Use the QR code to the right to access the PrairieLearn portion of this homework. **Please login using your Penn Google account.**

1. Most questions are designed to provide you with an infinite number of variations.
2. With these questions, if you respond incorrectly, you will have the opportunity to try again until you get the question right. To earn credit on the question, you must answer *any* variant from the first try.

## Exercise 1 – Generating Counter-Examples [ 2pts ]

Show that $\forall x,\ P(x) \vee \forall x Q(x)$ and $\forall x(P(x) \vee Q(x))$ are not logically equivalent. To make this interesting, come up with your own versions of predicates $P(x)$ and $Q(x)$ for which this does not hold.

## Exercise 2 – Direct Proof on Integers: Composite Sums [ 3pts ]

Let $n$ and $k$ be any positive integers such that $2 \leqslant k \leqslant n$. Show that (aka prove that) $n! + k$ is composite (that is the product of at least two primes).

## Exercise 3 – Statement on Primes [ 2pts ]

Prove the following: There is an integer $n$ such that $2n^2 - 5n + 2$ is prime

## Exercise 4 – Direct Proof on Integers: Odd Integers are Differences of Squares [ 4pts ]

Use a direct proof to show that every odd integer is the difference of two squares.

This means that the first line of your proof must read

**Consider $x$ to be an odd number.**

When we say the word "*square*" we mean a perfect square. This means numbers like $\{0, 1, 4, 9, 16, \ldots\}$.

More formally, an integer $n$ is said to be a square if and only if $\exists m \in \mathbb{Z}$ such that $m^2 = n$.

 

**Exercise 5 – Debugging a Proof** **[ 2pts ]**
Find the mistake in the following proof.

Importantly, please do remember that with a proof your first mistake in logic is your last mistake. Do not read any further once you have found that first mistake.

Theorem: The product of an even integer and an odd integer is even.

"*Proof: Suppose m is an even integer and n is an odd integer. Assume* $m \cdot n$ *is even, then by definition of even there exists an integer r such that* $m \cdot n = 2r$*. Also since* $m$ *is even, there exists an integer p such that* $m = 2p$*, and since n is odd there exists an integer q such that* $n = 2q + 1$*. Thus* $m \cdot n = (2p)(2q + 1) = 2r$*, where* $r$ *is an integer. By definition of even, then,* $m \cdot n$ *is even, as was to be shown.*"

**Exercise 6 – Proofs on Sets** **[ 12pts ]**

A. Prove that for any two sets $A$ and $B$, if $A \subseteq B$ and $B \subseteq A$, then $A = B$.

B. Prove that for any two sets $A$ and $B$, $A \cup B = B \cup A$.

C. Prove that for any three sets $A$, $B$, and $C$, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

D. Prove that for any set $A$, $A \cap \overline{A} = \emptyset$, where $\overline{A}$ is the complement of $A$. *You may need to use a different proof technique as in the previous questions.*

**Exercise 7 – Division is Closed Under Product and Affine Combinations of Integers** **[ 10pts ]**
Let $a$, $b$, $c$, $d$, and $x$, $y$, be integers.

Pick two out of the three following proof techniques: Direct proof, proof by contradiction, proof by contrapositive, proof by induction. Of these two techniques, use one to prove the first statement, and the other to prove the second statement.

A. If $d \mid a$ and $d \mid b$, then $d \mid (ax + by)$.

B. If $a \mid b$ and $c \mid d$, then $ac \mid bd$.

## Induction questions

**Exercise 8 – Induction Proof** **[ 4pts ]**
Show that $6|(n^3 - n)$ for all $n \in \mathbb{Z}^+$.

**Exercise 9 – Induction Proof** **[ 4pts ]**
Show that for $p \geqslant 5$ and $p \in \mathbb{Z}$, $4p < 2^p$.

**Exercise 10 – Induction Proof** **[ 4pts ]**
Show for $n > 2$, $n$ being a positive integer.

$$\left(1 - \frac{1}{2}\right) \cdot \left(1 - \frac{1}{3}\right) \cdots \left(1 - \frac{1}{n}\right) = \frac{1}{n}$$

## Exercise 11 – (Bonus) Tower of Hanoi [ 4pts ]

The Tower of Hanoi is a mathematical puzzle that consists of three pegs and a number of disks of different sizes which can slide onto any peg. The puzzle starts with the disks in a neat stack in ascending order of size on one peg, the smallest at the top, thus making a conical shape. The objective of the puzzle is to move the entire stack to another peg, obeying the following simple rules:

- Only one disk can be moved at a time.
- Each move consists of taking the upper disk from one of the stacks and placing it on top of another stack or on an empty peg.
- No disk may be placed on top of a smaller disk.

You can read more about the puzzle on HackerEarth, including implementation details to write a program to solve the puzzle.

Prove that the minimum number of moves required to solve a Tower of Hanoi puzzle with $n$ disks is $2^n - 1$.

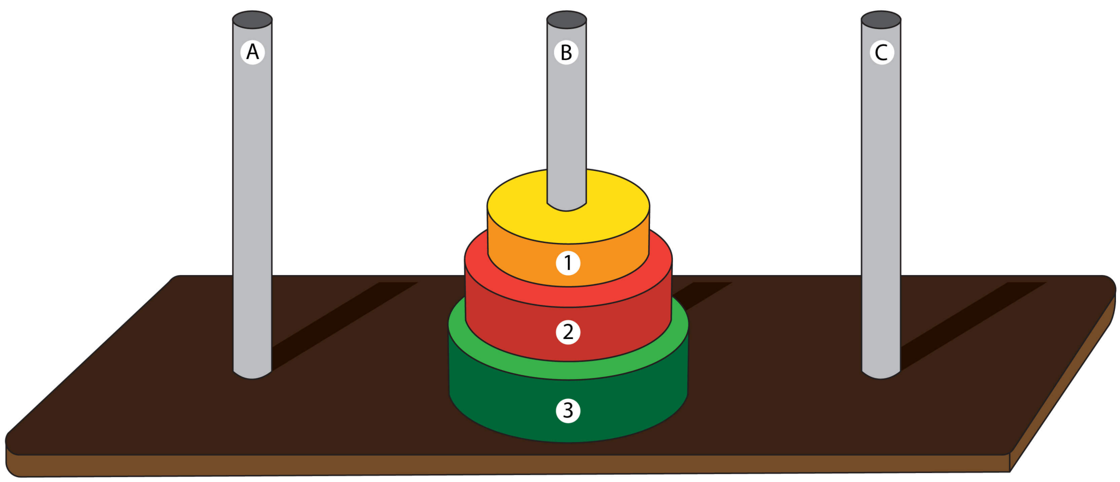


Figure 1: Illustration the Towers of Hanoi puzzle for $n = 3$ discs.

**Not graded** : The rest questions are not graded, only for practice

## Exercise 12 – Debug a Proof [ 4pts ]

A. Find the mistake in the following proof:

> **Theorem:** The sum of any two odd integers equals $4q + 2$ for some integer $q$.
> **Proof:** Suppose $m$ and $n$ are any two odd integers. By definition of odd, $m = 2q + 1$ for some integer $q$ and $n = 2q + 1$. By substitution, $m + n = 2q + 1 + 2q + 1 = 4q + 2$. Thus, the sum of any two odd integers equals $4q + 2$ for some integer $q$. This is what was to be shown.

B. Is the theorem in the previous part actually correct? If so, write a correct proof for it. If not, provide a counter example.

 

### Exercise 13 – Prove or Disprove [ 4pts ]

Prove or disprove, if $4$ divides an integer $n$, then $n + 2$ is not divisible by $4$.

### Exercise 14 – Proving an Implication [ 4pts ]

Prove that if $s^2 - 6s + 5$ is even, then $s$ is odd.

### Exercise 15 – Property of Indivisibility [ 4pts ]

Prove that if an integer $n$ does not divide $ab$ then it must be the case that $n$ does not divide $a$ and $n$ does not divide $b$.

### Exercise 16 – Proof by Contradiction on Sets [ 4pts ]

Prove that for all sets $A$ and $B$, if $B \subseteq \overline{A}$ then $A \cap B = \emptyset$. Do not write this proof using a Venn diagram. Please use contradiction instead.