CIT 5920–Mathematical Foundations of Computer Science
Homework 5: *Expectation and Logic*

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Complete the online portion on PrairieLearn. For the written section, use the **customized HW5 template on Overleaf**, shared on the course forum and Canvas. Each exercise should be on a separate page. Only submissions in this format will be accepted. Submit your written work on Gradescope by the deadline. For assistance or inquiries, don't hesitate to: Attend office hours; post questions on the class forum; ask about the motivation behind this material.

Guidelines:

- All guidelines from previous homeworks apply.
- In particular, write explanations unless you are explicitly told that we do not require them.

Exercise 0 – PrairieLearn Questions [20pts]

Use the QR code to the right to access the PrairieLearn portion of this homework. **Please login using your Penn Google account.**

1. Most questions are designed to provide you with an infinite number of variations.
2. With these questions, if you respond incorrectly, you will have the opportunity to try again until you get the question right. To earn credit on the question, you must answer *any* variant from the first try.

Exercise 1 – Contrapositive, Converse, Inverse—Oh My! [10pts]

In the following questions, it is not necessary to provide an explanation, as long as your answer is correct. However, if you are unsure about your answer, you can provide an explanation to receive partial credit.

A. Write the converse of the following statement "*If a number is even then it is divisible by 2*".

B. Write the contrapositive of the following statement "*A necessary condition for the ground to be wet is that it must be raining*".

C. Write the inverse of the following statement "*If a shape is a square, then it has four equal sides*".

D. Write the converse of the following statement "*If an integer is positive, then it is greater than zero*".

E. Write the contrapositive of the following statement "*For a set to be finite, it must have a limited number of elements*".

F. Write the inverse of the following statement "*If a number is prime, then it has only two distinct positive divisors*".

G. Each of the previous questions introduced an implication, and asked for the contrapositive, converse or inverse. Make a list of all the resulting propositions that have the same truth value as the original implication. Explain briefly.

Exercise 2 – Splitting Equivalence Statements [4pts]

In the following questions, it is not necessary to provide an explanation, as long as your answer is correct. However, if you are unsure about your answer, you can provide an explanation to receive partial credit.

A. Given the equivalence statement "*A figure is a rectangle* ***if and only if*** *it has four right angles*", split this into its two implication statements.

B. Given the equivalence statement "*A number is even* ***if and only if*** *it is divisible by 2*", split this into its two implication statements.

Exercise 3 – Evaluating Mathematical Logic Statements [5pts]

Are the following statements True or False? No explanation needed.

A. $\forall x \in \mathbb{Z}, x^2 > x$.

B. $\exists x \in \mathbb{R}, \frac{1}{x} = x$

C. $\forall x \in D \exists y \in F$ such that $xy \geq 4$ where $D = \{1, 2, 3\}$ and $F = \{3, 4\}$

D. $\exists x \in D \forall y \in F$ such that $xy \geq 4$ where D and F are the same sets defined above

E. $\exists y \in F \forall x \in D$ such that $xy \geq 5$ where D and F are the same sets defined above

Exercise 4 – Translating from English to Logic and Negation with Ease! [22pts]

Express the following English statements in logical form. That means defining one or more predicates, a domain of your variable(s), using ANDs, ORs, implications, quantifiers, etc.. **Remember that we want to see the original statement and the negated version.**

For example, given the statement "*All roses are red or fragrant*": We first define R as the set of roses. Then, we can write the statement as,

$$\forall x \in R, \text{IsRed}(x) \vee \text{IsFragrant}(x)$$

where $\text{IsRed}(x)$ is the predicate that is True if x is red, and $\text{IsFragrant}(x)$ is the predicate that is True if x is fragrant.

We can then negate this statement by applying the following process described in lecture:

1. **Change the Quantifier**: The first step in negating a universally quantified statement is to change the quantifier from "for all" ($\forall$) to "there exists" ($\exists$). This is based on the rule:

$$\neg \forall x P(x) \equiv \exists x \neg P(x)$$

2. **Apply De Morgan's Law**: De Morgan's Law states:

$$\neg(P \vee Q) \equiv \neg P \wedge \neg Q$$
$$\neg(P \wedge Q) \equiv \neg P \vee \neg Q$$

In our case, we have a disjunction (OR) inside the universal quantifier. So, when we negate it, we'll use the first of De Morgan's laws to change the OR to an AND and negate each of the predicates.

3. **Negate the Predicates**: The negation of the predicate $\text{IsRed}(x)$ is $\neg\text{IsRed}(x)$, which means "x is not red". Similarly, the negation of $\text{IsFragrant}(x)$ is $\neg\text{IsFragrant}(x)$, which means "x is not fragrant".

Negated Statement: Combining the steps above, the negation becomes:

$$\exists x \in R(\neg\text{IsRed}(x) \wedge \neg\text{IsFragrant}(x))$$

This translates to: "There exists a rose that is neither red nor fragrant."

A. If a real number is greater than 2 then its square is greater than 4.

B. Some cats are not lazy.

C. No dogs bark and bite simultaneously.

D. If a bird can fly, then it has wings.

E. No cars are both cheap and reliable.

F. Some books are neither boring nor long.

G. If a tree is old, it has deep roots.

H. For every student, there is another student who can help them with mathematics.

I. Some teachers teach the same subject.

J. Every book has an author who has written another book on the same topic.

K. Every student in a class respects every other student.

Exercise 5 – Evaluating Quantified Statements about Integers [6pts]

In this problem, the domain is the set of all integers. Which statements are true? If an existential statement is true, give an example. If a universal statement is false, give a counterexample.

A. $\exists x(x + x = 1)$

B. $\exists x(x + 2 = 1)$

C. $\forall x(x^2 - x \neq 1)$

D. $\forall x(x^2 - x \neq 0)$

E. $\forall x(x^2 > 0)$

F. $\exists x(x^2 > 0)$

Exercise 6 – Translating Mathematical Quantified Statements from English to Logic [4pts]

Consider the following statements in English. Write a logical expression with the same meaning. The domain is the set of all real numbers.

A. There is a number whose cube is equal to 2.

B. The square of every number is at least 0.

C. There is a number that is equal to its square.

D. Every number is less than or equal to its square plus 1.

Exercise 7 – Nuances on Quantified Statements [6pts]

In the following scenario, the domain is a set of individuals who aspire to become entrepreneurs. Let's denote one of these individuals as Alex. Define the following predicates:

- $I(x)$: x has an innovative idea
- $C(x)$: x has the capital to start a business
- $T(x)$: x has undergone entrepreneurship training

A. At least one individual has an innovative idea.

B. Everyone has the capital and has undergone training.

C. Everyone with an innovative idea also has the capital.

D. There exists an individual who has an idea and has undergone training.

E. Everyone who does not have capital has undergone training.

F. Everyone without training either has an idea or the capital (or both).

G. Some individual without training and capital has an innovative idea.

H. Each individual without an idea has either undergone training or has the capital (or both).

I. Alex has an innovative idea but does not have the capital.

J. Someone other than Alex has undergone training.

K. Everyone except Alex has an innovative idea.