CIT 5920
Recitation 8

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Penn Engineering

Overview for Today

- Logistics
- Question 1
- Question 2
- Question 3
- Question 4
- Question 5
- Question 6
- Question 7

Logistics

- HW5 will be released soon

Propositions and Predicates

Propositions: Complete statements that are definitely true or false (no variables).

Predicates: Statements involving variables that become propositions when specific values are assigned to those variables.

Propositions and Predicates Examples

Propositions:

1. "All cats are mammals." (This is a true proposition.)
2. "Water boils at 50 degrees Celsius at 1 atm." (This is a false proposition.)

Predicate:

P(x): x is a multiple of 3

(This becomes a proposition when we assign a value to x. For example, if x = 9, P(x) is true. If x = 10, P(x) is false.)

Boolean Variables

Conjunction operator (AND) is denoted by $\wedge$ (\wedge in LaTeX)

Disjunction operator (OR) is denoted by $\vee$ (\vee in LaTeX)

NOT is denoted by $\neg$ (\neg in LaTeX)

De-Morgan's Law for Booleans

$$\neg(p \lor q) = \neg p \land \neg q$$

$$\neg(p \land q) = \neg p \lor \neg q$$

If and Only If

P if and only if (iff) Q means $P \rightarrow Q$ and $Q \rightarrow P$ ($P \leftrightarrow Q$)

$P \rightarrow Q$

- Q if P
- P only if Q

Quantifiers

Universal quantifier: $\forall$ (for all)

Example: $\forall x \in \mathbb{R},\ x^2 \geq x$

Existential quantifier: $\exists$ (there exists)

Example: $\exists x \in \mathbb{R}^+,\ x + 1 > 2$

Negation of Quantifiers

$$\neg \forall x \; P(x) \equiv \exists \; \neg P(x)$$
$$\neg \exists x \; P(x) \equiv \forall \; \neg P(x)$$

Contrapositive, Converse, and Inverse

There are 3 statements that are closely related to $p \rightarrow q$.

- Contrapositive: $\neg q \rightarrow \neg p$
- Converse: $q \rightarrow p$
- Inverse: $\neg p \rightarrow \neg q$

Contrapositive is equivalent to the original statement $p \rightarrow q$.
Converse and inverse are equivalent (inverse is the contrapositive of converse).

Example

- $P \rightarrow Q$: If n is even, then n^2 is even.
- Contrapositive ($\neg Q \rightarrow \neg P$): If n^2 is odd, then n is odd.
- Converse ($Q \rightarrow P$): If n^2 is even, then n is even.
- Inverse: ($\neg P \rightarrow \neg Q$): If n is odd, then n^2 is odd.

Contrapositive with DeMorgan's Laws

Ex. If $x \cdot y = 0$, then $x = 0$ or $y = 0$.

$\neg Q = \neg\ (x = 0 \text{ or } y = 0)$

$= x \neq 0 \text{ and } y \neq 0$ (by DeMorgan's Laws)

If $x \neq 0$ and $y \neq 0$, then $x \cdot y \neq 0$

AND vs. Implication

Use AND ($\wedge$) when both conditions need to be true simultaneously.

Use implication ($\rightarrow$) when one statement's truth relies on or guarantees the truth of another.

Example:

English: "All birds can fly and sing."

Let B(x) be the predicate "x is a bird," F(x) be the predicate "x can fly," and S(x) be the predicate "x can sing."

Logic: $\forall x(B(x) \rightarrow (F(x) \wedge S(x)))$

This sentence uses **implication** because being a bird implies certain abilities.

Within the implication, we use **AND** because both abilities (flying and singing) must hold for every bird.

BUT The conditional statement $P \rightarrow Q$ is logically equivalent to $\neg P \vee Q$

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Direct Proof Example

Theorem: The sum of two odd integers is even.

Proof:

1. We show that if x and y are odd integers, then x + y is even.
2. Let x and y be any odd integers.
3. Since x is odd, by the definition of odd integers, we know that $x = 2m + 1$ for some integer m.
4. Similarly, since y is odd, there exists an integer n such that $y = 2n + 1$.
5. Now observe that: $x + y = (2m + 1) + (2n + 1) = 2m + 2n + 2 = 2(m + n + 1)$.
6. Since $m + n + 1$ is an integer (because the sum of integers is an integer), let $a = m + n + 1$.
7. Then we can write $x + y = 2a$.
8. Therefore, $2 \mid (x + y)$, meaning x + y is divisible by 2.
9. Thus, x + y is even.

Question 1

Write the following statement using logic symbols, predicates, etc. We want you to use the domain of all humans.
Be careful about the uniqueness part

Every MCIT student has a unique (only 1) PennID

Don't worry about two people possibly having the same PennID here. By unique, in this problem we mean that each MCIT student has EXACTLY one PennID, not more than one.

Answer 1

Solution: Let H be the set of all humans and S be the set of all possible PennIDs. For $x \in H$ and $y \in S$, let $M(x)$ denote "x is an MCIT student," and $ID(x, y)$ denote "x has PennID y."
Then using these symbols and predicates, we can encode the message as:

$$\forall x \in H\,(M(x) \to (\exists y \in S\ \mathrm{ID}(x,y) \wedge \nexists z \in S\ \ (\mathrm{ID}(x,z) \wedge z \neq y)))$$

Question 2

Write the following theorem as mathematically as possible. Use quantifiers, the symbols for the fundamentals sets etc.

If a and b are integers with a being positive, there exist unique integers q and r such that $b = qa + r$ where r, the remainder, will be between 0 and a (inclusive).

Side question: Is this true? Just say yes or no.

Answer 2

Solution: $\forall a \in \mathbb{Z}^+ \; \forall b \in \mathbb{Z}, (\exists q, r \in \mathbb{Z}, (0 \leqslant r \leqslant a) \wedge (b = aq + r) \wedge (\forall s, t \in \mathbb{Z}, (s \neq q) \vee (t \neq r) \implies b \neq sa + t))$
The last part is expressing the uniqueness, the fact that no pair aside from q and r can produce the result.

Side question: It isn't true because of the uniqueness part and since r can equal a. Consider a=5, b=10 Then both q=2, r=0 and q=1, r=5 work. If r was strictly less than a, then this would be true.

Question 3

Check if this reasoning is logical by writing each of the statements using logic and quantifiers. Then show why it is the case that the final statement follows from the previous ones.

All whales are mammals. Some whales are carnivorous. All carnivorous organisms eat other animals. Therefore, some mammals eat other animals.

Answer 3

Solution: Let x be any animal. We define the predicates below:

$W(x)$: x is whale

$M(x)$: x is mammal

$C(x)$: x is carnivorous

$E(x)$: x eats other animals

Then the statements are:

All whales are mammals: $\forall x, W(x) \implies M(x)$

Some whales are carnivorous: $\exists x, W(x) \wedge C(x)$

All carnivorous organisms eat other animals: $\forall x, C(x) \implies E(x)$

Now we can combine the first predicate with second to see that $\exists x, M(x) \wedge C(x)$

Now because of the third predicate, we can conclude that $\exists x, M(x) \wedge E(x)$

Question 4

Prove that an integer is odd if and only if it is the sum of two consecutive integers.

Answer 4

Solution: To prove if and only if (iff) statements, we must prove both directions of the implication.

If an integer is odd, then it is the sum of two consecutive integers. ($\Longleftarrow$)

Proof. Let $x \in \mathbb{Z}$ and x be odd.
Then by definition, $\exists a \in \mathbb{Z}$ such that $x = 2a + 1$. Then

$$\begin{aligned} x &= 2a + 1 \\ &= a + a + 1 \\ &= a + (a + 1) \end{aligned}$$

Which is the sum of two consecutive integers, a and $a + 1$.

If an integer is the sum of two consecutive integers, then it is odd. ($\Longrightarrow$)

Proof. Let $x \in \mathbb{Z}$ such that x is the sum of two consecutive integers.
Therefore, $\exists a \in \mathbb{Z}$ such that $x = a + (a + 1)$. Then

$$\begin{aligned} x &= a + (a + 1) \\ &= a + a + 1 \\ &= (a + a) + 1 \\ &= 2a + 1 \end{aligned}$$

Therefore, x is odd by definition.

Question 5

Prove that the difference between two consecutive perfect squares will always be odd. Consecutive perfect squares means the squares of two integers that are consecutive. For example, 144 and 169 are consecutive perfect squares.

Answer 5

Solution: Let n^2 and $(n+1)^2$ be any two consecutive perfect squares for some arbitrary integer n. Then $(n+1)^2 - n^2 = (n^2 + 2n + 1) - n^2 = 2n + 1$. Therefore, by definition this is odd (since n is an integer here).
(We can also similarly see that $n^2 - (n+1)^2 = -(2n+1)$, which is also odd by definition.)

See you next week!