CIT 5920–Mathematical Foundations of Computer Science
Midterm 1: *Sets, Relations, Functions, and Combinatorics*

Version: September 11, 2024

Exam Instructions

- **PLEASE DO NOT BEGIN UNTIL EXPLICITLY INSTRUCTED TO DO SO.**
- **Leaving Early:** To ensure everybody can take the exam in a quiet environment, we ask that you remain in the room until the end of the exam. If you finish early, you can use the remaining time to review your answers.
- **Anonymity:** To ensure fairness in grading, please refrain from writing your name on the exam. We will grade papers anonymously, but rest assured that your final grade will be recorded accurately.
- **Electronic Devices:** Please set your cellphones to "Do Not Disturb" mode. Note that the use of calculators is not permitted during this exam.
- **Simplification:** It's okay to leave your answers in an unsimplified form. For instance, you don't need to simplify "2 + 2". We're assessing your understanding, not your mental arithmetic skills.
- **Answer Space:** Ensure you provide your answers within the designated spaces. We won't be checking the back of any page.
- **Scratch Paper:** There's a blank sheet provided at the end of the exam for any rough work. Additionally, feel free to use the back of any sheet if you need more space.
- **Time Management:** Some questions might be more challenging than others. We recommend not spending too much time on any single question to ensure you attempt all of them.
- **Explanations:** Unless a question states "No explanation needed", please provide a brief rationale for your answer.

Your **PennID** (the 8 **digits** in big font on your penncard): ______________________________

The PrairieLearn portion will be optional (if you do not do it, we will just weigh your written portion more to compensate), will be released after both recitations today are over, and will be due at 11:59pm on Thursday, October 12th.

GOOD LUCK!

Exercise 1 – Mathematical Notation [20pts]

A. Let A, B, S be sets. Translate the following statements using the best mathematical notation:

(i) "The intersection of A and B is not empty."
(ii) "The set A contains the element 3."
(iii) "The set containing just 3 is a subset of A."
(iv) "The set B does not contain the element 3."
(v) "The set A from which we remove the element 3 is equal to the set B."
(vi) "The union of A and its complement is the universe."
(vii) "The union of A and B is the set S."
(viii) "The set A is a subset of all integers."

B. Write the following sets, which are expressed using set builder notation, using set roster notation:

(i) $\{3m \mid m \in \mathbb{Z} \text{ and } 10 < m < 15\}$.
(ii) $\{a \mid a \in \mathbb{Z} \text{ and } a^2 \in \mathbb{Z} \text{ and } 4 \leqslant a < 5\}$.
(iii) $\{3y^2 + 12 \mid y \in \mathbb{Z} \text{ and } -2 < y < 3\}$.
(iv) $\{x \mid x \in \mathbb{R} \text{ and } x \in \mathbb{Z}_4 \text{ and } 5 \leqslant x \leqslant 23\}$ (where $\mathbb{Z}_4$ is the set of integers that are multiples of 4).

C. Let $C = \{1, x\}$, what does the expression $|\mathcal{P}(C)|$ represent (explain using words)? What is the value of this expression?

D. Let $D = \{6, 7, 8\}$ and $E = \{a, b, c\}$.

(i) What is $D \times E$ (using words)?
(ii) What is the value of $|D \times E|$?
(iii) Is $\{a, 8\}$ an element of $D \times E$? Why or why not?
(iv) Is $\{a, 8\}$ an element of $E \times D$? Why or why not?
(v) Provide two examples of elements of $D \times E$.

Exercise 2 – Set Identity [2pts]

Show using the right relations: $A \cup (B \setminus C) = (A \cup B) \setminus (C \setminus A)$.

Exercise 3 – Reflexive Relations [2pts]

A student makes the following statement:

"*On a set of 3 elements, it is impossible to have a relation that has at least 2 elements such that it is symmetric and transitive but not reflexive.*"

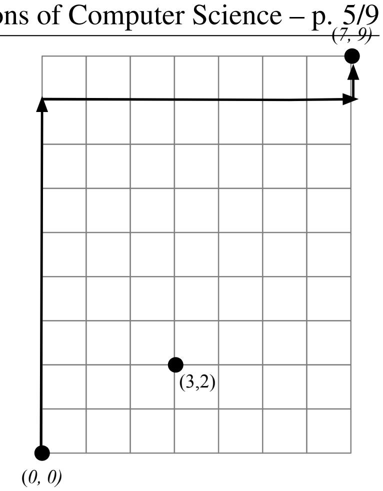
Is the student right? If they are right, provide some brief justification. If they are wrong, provide a counter-example.

Exercise 4 – Symmetric Relations [6pts]

A. On the set $\{1, 2, 3\}$ is the relation $\{(1, 1), (2, 2), (3, 3)\}$ symmetric? Just answer yes or no. No explanation needed.

B. Consider a set A that has n elements. How many distinct symmetric relations can be defined on A? Please provide an explanation.

Exercise 5 – Counting Monotonic Paths in a Grid [4pts]



Consider a grid that is 7 units wide and 9 units tall. A *monotonic path* is one that begins at point $(0, 0)$ (the bottom left corner) and traverses to $(7, 9)$ (the top right corner), using only upward ($\uparrow$) and rightward ($\rightarrow$) moves. For example, the path on the right is a monotonic path:

$$(\uparrow,\uparrow,\uparrow,\uparrow,\uparrow,\uparrow,\uparrow,\uparrow,\rightarrow,\rightarrow,\rightarrow,\rightarrow,\rightarrow,\rightarrow,\rightarrow,\uparrow).$$

A. How many different monotonic paths are possible?

B. How many such monotonic paths go through the point $(3, 2)$?

Exercise 6 – Picking Teams [6pts]

A. There are three different fields for the students to practice on. How many different ways are there to assign the 27 players to the 3 fields in teams of 9—knowing that only 1 team can ?

B. How many ways are there to assign the 27 players to 3 teams of 9, without regard for which team is on which field?

C. How many ways are there to assign the 27 players to 3 teams of 9, and for each team to choose 1 of its players as captain? As in the previous question, we do not care which team is on which field.

D. One of the teams plays 10 games against teams from other schools, ending the season with a 7-3 record (they won 7 games out of the 10 games they played, and lost 3 games). How many different sequences of wins and losses could have led to this outcome?

Exercise 7 – Counting Solutions [7pts]

How many non-negative ($x_i \geqslant 0$) integer solutions exist for the following systems of equations or inequations (remember that a system means that all of these need to be satisfied simultaneously)?

A.

$$x_1 + x_2 + x_3 + x_4 = 21$$

B.

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &\leqslant 21 \\ x_2 &\geqslant 2 \\ x_3 &\geqslant 3 \\ x_4 &\leqslant 19 \end{aligned}$$

C.

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &\leqslant 21 \\ x_2 + x_3 &= 5 \end{aligned}$$

Exercise 8 – Probability **[2pts]**

There are 15 oranges that are being thrown randomly into 6 baskets. Each basket has enough capacity to hold all 15 oranges. What is the probability that the oranges don't all fall in the same basket? For the purposes of this problem, think of the oranges as distinct and the baskets as distinct.

Compendium of Formulas

- The number of ways to choose k items out of n is
$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$
- The number of ways to pick and arrange k items out of n is $P(n,k) = n!/k!$.
- $|P(A)| = 2^{|A|}$.
- Number of ways to arrange n items out of which k_1 are identical of one kind and k_2 are identical of another kind is.
$$\frac{n!}{k_1!k_2!}$$

Properties of Relations

- A relation is *symmetric* if and only for every a that is related to b, b is related to a.
- A relation on set X is *reflexive* if and only if for every $x \in X$, (x,x) is in the relation.
- A relation R is *transitive* if and only if aRb and bRc gives us aRc.
- A relation is antisymmetric if aRb and bRa only happens when a and b are equal.

Set Identities

Name	Identities	
Idempotent laws	$A \cup A = A$	$A \cap A = A$
Associative laws	$(A \cup B) \cup C = A \cup (B \cup C)$	$(A \cap B) \cap C = A \cap (B \cap C)$
Commutative laws	$A \cup B = B \cup A$	$A \cap B = B \cap A$
Distributive laws	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Identity laws	$A \cup \emptyset = A$	$A \cap U = A$
Domination laws	$A \cap \emptyset = \emptyset$	$A \cup U = U$
Double complement law	$\overline{\overline{A}} = A$	
Complement laws	$A \cap \overline{A} = \emptyset$ $\overline{U} = \emptyset$	$A \cup \overline{A} = U$ $\overline{\emptyset} = U$
De Morgan's laws	$\overline{A \cup B} = \overline{A} \cap \overline{B}$	$\overline{A \cap B} = \overline{A} \cup \overline{B}$
Absorption laws	$A \cup (A \cap B) = A$	$A \cap (A \cup B) = A$