CIT 5920
Recitation 6

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Penn Engineering

Overview for Today

- Logistics
- Probability Review
- Question 1 (4 minutes)
- Question 2 (5 minutes)
- Question 3 (3 minutes)
- Question 4 (5 minutes)
- Question 5 (4 minutes)
- Question 6 (4 minutes)

Logistics

- HW4 is due 11:59 PM on Monday, October 28
- Midterm 1 grades will be released next week

Addition Rule

Events A and B are **mutually exclusive** if

$$A \cap B = \emptyset$$

For mutually exclusive events:

$$P(A \cup B) = P(A) + P(B)$$

Addition Rule Example

Consider drawing a single card from a standard deck of cards. What's the probability of drawing a heart or a spade?

- Event A - Drawing a heart
- Event B - Drawing a spade

$$P(A) = \frac{\text{Number of hearts}}{\text{Total number of cards}} = \frac{13}{52} = \frac{1}{4}$$

$$P(B) = \frac{\text{Number of spades}}{\text{Total number of cards}} = \frac{13}{52} = \frac{1}{4}$$

Since they are mutually exclusive,

$$P(A \cup B) = P(A) + P(B) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

Complement Rule

The complement of an event A, (A', $\overline{A}$) consists of **all** outcomes **not** in A

$$P(\overline{A}) = 1 - P(A)$$

Complement Rule Example

What is the probability of getting at least one head in 3 coin flips?

You could calculate it as:

$$P(\text{one head}) = \binom{3}{1}(\frac{1}{2})^1(\frac{1}{2})^2 = 3 \times \frac{1}{8} = \frac{3}{8}$$

$$P(\text{two heads}) = \binom{3}{2}(\frac{1}{2})^2(\frac{1}{2})^1 = 3 \times \frac{1}{8} = \frac{3}{8}$$

$$P(\text{three heads}) = \binom{3}{3}(\frac{1}{2})^3(\frac{1}{2})^0 = 1 \times \frac{1}{8} = \frac{1}{8}$$

$$P(\text{at least one head}) = P(\text{one head}) + P(\text{two heads}) + P(\text{three heads}) = \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = \frac{7}{8}$$

(But what if there are 100 coin flips?)

$$P(\text{at least one head}) = 1 - P(\text{all tails})$$

$$1 - P(\text{all tails}) = 1 - (\frac{1}{2})^3$$

Experiment, Sample Space, Events, and Probability

Experiment: A procedure that yields one out of several possible outcomes. It's an action or process where the outcome is uncertain

Sample Space (S): the set of all possible outcomes of an **experiment**

Event: a subset of the sample space. An event occurs if the outcome of the experiment is an element of the event set.

P[E_i] (*probability that event E_i occurs*) = |**E_i**| / |**S**|

(Positive outcomes)/ (Total outcomes)

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Example

Imagine a game where players spin a color wheel divided into seven equal sections, colored: red, orange, yellow, green, blue, indigo, violet.

Identify the experiment, the sample space, and some events.

Example Solutions

Experiment: Spinning the color wheel once and observing which color the pointer lands on.

Sample Space (S): The sample space consists of all possible outcomes when spinning the wheel. In this case: S = {red, orange, yellow, green, blue, indigo, violet}

Events: We can define various events based on this sample space. Here are a few examples: a) Event A: The pointer lands on a primary color. A = {red, yellow, blue} b) Event B: The pointer lands on a color starting with the letter 'v'. B = {violet} c) Event C: The pointer lands on a warm color. C = {red, orange, yellow}

Example

Imagine a game where players spin a color wheel divided into seven equal sections, colored: red, orange, yellow, green, blue, indigo, violet.

i) What is the probability of Event A (landing on a primary color?)

ii) If we spin the wheel twice, what is the probability of getting orange on the first spin, and green on the second?

iii) What is the probability that the pointer does not land on indigo?

Example Solutions

i) Probability of landing on a primary color is 3/7

ii) the probability of landing on orange and then green is (1/7) x (1/7) = (1/49)

iii) The probability of landing not on indigo is (6/7). We can also say it's 1- P(indigo), or 1 - (1/7)

Inclusion - Exclusion Principle

Two events:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Three events:

$$\begin{aligned} P(A \cup B \cup C) = {} & P(A) + P(B) + P(C) \\ & - P(A \cap B) - P(A \cap C) - P(B \cap C) \\ & + P(A \cap B \cap C) \end{aligned}$$

Conditional Probability

The conditional probability is the probability that event A will occur given that event B has occurred.

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Conditional Probability Examples

What is the probability of the card drawn is a queen given it is a spade?

- Event A - Drawing a queen
- Event B - Drawing a spade
- P(A∩B) is the probability of drawing the queen of spades
- P(B) is the probability of drawing a spade

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{52}}{\frac{13}{52}} = \frac{1}{13}$$

Independence

$$P(A \cap B) = P(A) \times P(B)$$

$$P(A|B) = P(A)$$

Question 1

Exercise 1
A box contains 100 items of which 4 are defective. Two items are chosen at random from the box. What is the probability of selecting:

A. 2 defectives if the first item is not replaced?

B. 2 defectives if the first item is put back before choosing the second item;

C. 1 defective item and 1 non-defective item (in no particular order) if the first item is not replaced?

Consider both orders

Answer 1 (cont.)

A: 

4 in 100

x

3 in 99

Answer 1 (cont.)

A. 2 defectives if the first item is not replaced?

Solution: $\frac{\binom{4}{2}}{\binom{100}{2}}$

The total number of ways to choose 2 items out of 100 items is $\binom{100}{2}$. Out of those, there are $\binom{4}{2}$ ways to select 2 defective items out of the 4 defective items that exist.

Alternate approach: $\frac{4}{100} * \frac{3}{99}$

There is a 4 in 100 chance that the first item is defective, and then, since it is not replaced, we have 99 items left of which 3 are defective, so there is a 3 in 99 chance that the next item is defective.

Answer 1 (cont.)

A:

x

4 in 100 3 in 99

B:

x

4 in 100 4 in 100

Answer 1 (cont.)

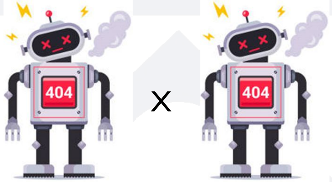
B. 2 defectives if the first item is put back before choosing the second item;

Solution: $\frac{4^2}{100^2}$

There is a 4 in 100 chance that the first item is defective, and then, since it replaced, we once again have 100 items, 4 of which are defective, so there is a 4 in 100 chance that the second item is defective.

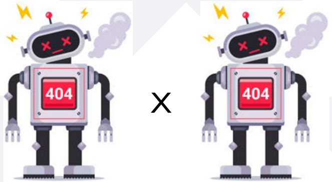
Answer 1

A:



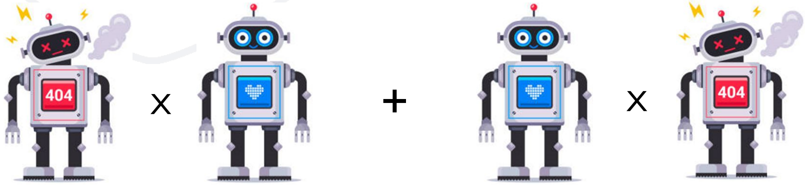
4 in 100 x 3 in 99

B:



4 in 100 x 4 in 100

C:



4 in 100 x 96 in 99 + 96 in 100 x 4 in 96

Answer 1 (cont.)

C. 1 defective item and 1 non-defective item (in no particular order) if the first item is not replaced?

Solution: $\frac{\binom{4}{1} * \binom{96}{1}}{\binom{100}{2}}$

Denominator has ½ by combination formula

The total number of ways to choose 2 items out of 100 items is $\binom{100}{2}$. To choose a defective item and then a non-defective one, we would have to choose 1 out of the 4 defective items, and then 1 out of the 96 non-defective items.

Alternate approach: $2 * \frac{4}{100} * \frac{96}{99}$

There is a 4 in 100 chance that the first item is defective, and then, since it is not replaced, we have 99 items left of which 96 are not defective, so there is a 96 in 99 chance that the next item is defective. We could also have the reverse case where the first item has a 96 in 100 chance of being non-defective, and then the second item has a 4 in 99 chance of being defective, which is why we multiply by 2 overall.

Question 2

We are going to ask a random student about their birthday month.

What is the sample space?

What is the probability that they are

i) born in a month with 31 days

ii) born in a month starting with a J?

iii) born in a month starting with J or with 31 days?

Question 2

What is the sample space? The set of all possible months S = {Jan, Feb, Mar, Apr, May, Jun, Jul, Aug, Sep, Oct, Nov, Dec}

i) there are 7 months with 31 days. So 7/12
ii) there are 3 months that start with J, so 3/12 or ¼
iii) (7/12) + (3/12) - (2/12)

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Question 2

Exercise 2

How many 10 bit numbers are there with no 2 consecutive ones? For 10 bit numbers, we allow the most significant bit to be 0.

Hint: Think about each case for the numbers of ones you'll have and how many positions they can be placed in!

_0_0_0_0_0_0_

Answer 2

Exercise 2
How many 10 bit numbers are there with no 2 consecutive ones? For 10 bit numbers, we allow the most significant bit to be 0.

Solution: No 2 consecutive ones can be broken in the following cases:

(i) No ones at all. 1 way.

(ii) A single one. 10 ways (just pick one of the 10 digits to be a one).

(iii) 2 ones. That is 8 zeros in the form 0 0 0 0 0 0 0 0. The ones can go into any of the gaps. That means $\binom{9}{2}$.

(iv) 3 ones. That is 7 zeros in the form 0 0 0 0 0 0 0. The ones can go into any of the gaps. That means $\binom{8}{3}$.

(v) 4 ones. That is 6 zeros. So 7 gaps which means $\binom{7}{4}$.

(vi) 5 ones. That is 5 zeros. So 6 gaps which means $\binom{6}{5}$.

If we went any further we would only have 5 gaps in which to fill 6 ones, which wouldn't work.
To have 6 or more ones in a 10 digit binary number would require us to have consecutive ones.

Add all cases together since they are mutually exclusive, and we get: $1 + 10 + \binom{9}{2} + \binom{8}{3} + \binom{7}{4} + \binom{6}{5} = 144$ ways.

Question 3

In a die-coin experiment, a standard, fair six-sided die is rolled and then a fair coin is tossed the number of times showing on the die.

Let N denote the outcome on the die and H be the event that all coin tosses result in heads.

a) Find P(H)

b) Find P(N = 5 | H)

Answer 3

a) $N = [1, 6]$ and the probability for each event is $\frac{1}{6} \times \left(\frac{1}{2}\right)^N$:

$$P(H) = \frac{1}{6}\left(\left(\frac{1}{2}\right)^1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^5 + \left(\frac{1}{2}\right)^6\right) \approx 0.164$$

b)

$$P(N = 5 \mid H) = \frac{P(N = 5 \cap H)}{P(H)} = \frac{\frac{1}{6} \times \left(\frac{1}{2}\right)^5}{0.164} \approx 0.032$$

Question 3

Exercise 3

Provide a single counterexample to the following statement: If Q and R are relations on S, and both of them are transitive, then $Q \cup R$ is also going to be transitive. To be clear, a single counterexample means one single example with an actual set S and two actual relations Q and R such that this statement does not work out.

Answer 3

Exercise 3

Provide a single counterexample to the following statement: If Q and R are relations on S, and both of them are transitive, then $Q \cup R$ is also going to be transitive. To be clear, a single counterexample means one single example with an actual set S and two actual relations Q and R such that this statement does not work out.

Solution: The easiest way to do this is to come up with vacuously true transitive relations and then show their union messes it up. Take these two transitive relations:

$$\{(1,2)\}$$

$$\{(2,3)\}$$

They are vacuously true transitive relations: they never violate the transitive requirement, but don't really have a great example of transitivity in them. In other words, they are transitive because we cannot show that they are not transitive.

However, their union is: $\{(1,2),(2,3)\}$ which is not transitive because the tuple $(1,3)$ is missing.

Question 4

Exercise 4

Simplify $\overline{(A \cap B)} \cap \overline{B} \cap A$ into the simplest possible form. State all the rules being used in the simplification.

Name	Identities	
Idempotent laws	$A \cup A = A$	$A \cap A = A$
Associative laws	$(A \cup B) \cup C = A \cup (B \cup C)$	$(A \cap B) \cap C = A \cap (B \cap C)$
Commutative laws	$A \cup B = B \cup A$	$A \cap B = B \cap A$
Distributive laws	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Identity laws	$A \cup \emptyset = A$	$A \cap U = A$
Domination laws	$A \cap \emptyset = \emptyset$	$A \cup U = U$
Double complement law	$\overline{\overline{A}} = A$	
Complement laws	$A \cap \overline{A} = \emptyset$ $\overline{U} = \emptyset$	$A \cup \overline{A} = U$ $\overline{\emptyset} = U$
De Morgan's laws	$\overline{A \cup B} = \overline{A} \cap \overline{B}$	$\overline{A \cap B} = \overline{A} \cup \overline{B}$
Absorption laws	$A \cup (A \cap B) = A$	$A \cap (A \cup B) = A$

Answer 4

Exercise 4

Simplify $\overline{(A \cap B)} \cap \overline{B} \cap A$ into the simplest possible form. State all the rules being used in the simplification.

Solution:

$$
\begin{aligned}
\overline{(A \cap B)} \cap \overline{B} \cap A &= (\overline{A} \cup \overline{B}) \cap \overline{B} \cap A && \text{(De-Morgan's Law)} \\
&= ((\overline{A} \cap \overline{B}) \cup (\overline{B} \cap \overline{B})) \cap A && \text{(Distributive)} \\
&= ((\overline{A} \cap \overline{B}) \cap A) \cup (\overline{B} \cap A) && \text{(Distributive)} \\
&= ((\overline{A} \cap A) \cap (\overline{B} \cap A)) \cup (\overline{B} \cap A) && \text{(Distributive)} \\
&= ((\emptyset) \cap (\overline{B} \cap A)) \cup (\overline{B} \cap A) && \text{(Domination Law)} \\
&= (\emptyset) \cup (\overline{B} \cap A) && \text{(Domination Law)} \\
&= \overline{B} \cap A && \text{(Identity)}
\end{aligned}
$$

Question 5

Exercise 5

Consider the function: $f(n) = \lfloor \sqrt{n} \rfloor$ where we have both the domain and co-domain set to be $\mathbb{N}$.
The floor symbol truncates the decimal part $\lfloor 2.56 \rfloor = 2$. You can consider this to be similar to how floor division works in programming.

A. Is the function f one-to-one? Why or why not?

B. Is this same function f onto? Why or why not?

Answer 5

A. Is the function f one-to-one? Why or why not?

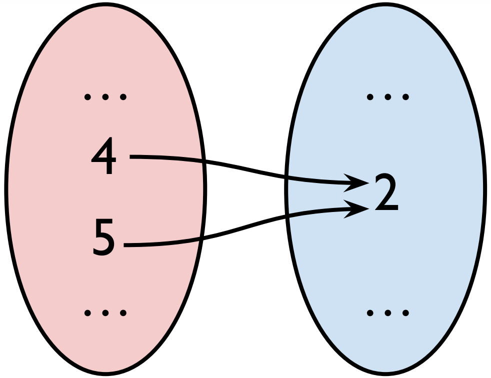
Solution: This function is not one-to-one.

$f(4) = \lfloor \sqrt{4} \rfloor = \lfloor 2 \rfloor = 2$

But now consider $f(5) = \lfloor \sqrt{5} \rfloor$

Since $\sqrt{5} < 3$, the floor function will take it down to 2, i.e. $f(5) = \lfloor \sqrt{5} \rfloor = 2$

This means that both $f(4)$ and $f(5)$ will map to 2, which means that this function cannot possibly be one-to-one.



Answer 5 (cont.)

B. Is this same function f onto? Why or why not?

Solution: This function is onto.

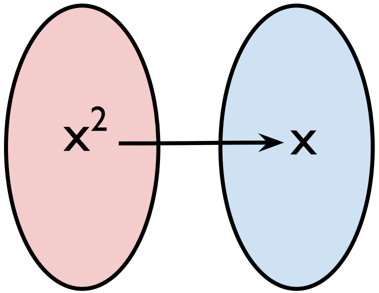
Proving it is onto:

Let's take x to be an arbitrary element.

We want to show that every $x \in \mathbb{N}$ will be mapped onto.

For every x, we can can choose an x^2 such that:

$f(x^2) = \lfloor \sqrt{x^2} \rfloor = x$. We just showed that $f(x^2) = x$, which means that for every $x \in \mathbb{N}$, we can choose a number (x^2) that can map onto it. So this function must be onto.



For all x

Question 4

A deck of cards contain 52 cards: 26 red and 26 black. If a card is drawn at random and found to be red, what is the probability that it is a heart (given that there are 13 hearts and 13 diamonds, both red)?

Answer 4

Given: total cards = 52, red cards = 26 (13 hearts + 13 diamonds), black cards = 26

To Find: Probability that the card is a heart given that it is red, i.e., $P(\text{Heart}|\text{Red})$.

Solution:

Step 1: Define the events. Let H be the event that the card drawn is a heart. Let R be the event that the card drawn is red.

Step 2: Use the formula for conditional probability.

$$P(H|R) = \frac{P(H \cap R)}{P(R)}$$

Step 3: Find $P(H \cap R)$. The probability that a card is both a heart and red is simply the probability that it's a heart (since all hearts are red). There are 13 hearts in a deck of 52 cards.

$$P(H \cap R) = \frac{13}{52} = \frac{1}{4}$$

Step 4: Find $P(R)$. The probability that a card is red (either a heart or a diamond) is:

$$P(R) = \frac{26}{52} = \frac{1}{2}$$

Step 5: Plug in the values into the conditional probability formula.

$$P(H|R) = \frac{P(H \cap R)}{P(R)} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{4} \times 2 = \frac{1}{2}$$

Conclusion: Given that a card drawn is red, the probability that it is a heart is $\frac{1}{2}$ or 50%.

Question 5

You roll two fair 6-sided dice (Die 1 and Die 2), and you flip a fair coin.
Define the following events:

- A: The sum of the two dice is 7.
- B: The outcome of the coin flip is heads.
- C: Both dice show the same number (a double).

Determine whether the events A and B are independent, and whether the events A and C are independent.

Answer 5

Probability of Event A (Sum of dice is 7):

Events = {(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)}

P (A) = 6 / 36 = 1 / 6 .

Probability of Event B (Coin flip is heads):

P (B) = 1/2 .

Probability of Event C (Both dice show the same number):

P (C) = 6/36 = 1/6

Answer 5

For the dice rolls, there are 6×6=36 possible outcomes (since each die has 6 faces).

The possible outcomes of the coin flip are heads (H) and tails (T).

Thus, there are 36×2=72 total outcomes when combining the dice rolls and the coin flip.

EX. {(1,1), H} means first and second roll is 1 and the coin is head.

These are the 6 outcomes where the sum of the dice is exactly 7. (previous slides).

Therefore, the probability $P(A \cap B)$ is:

$$P(A \cap B) = \frac{6}{72} = \frac{1}{12}$$

$$P(A \cap B) = P(A) \cdot P(B) = \frac{1}{6} \cdot \frac{1}{2} = \frac{1}{12}.$$

Thus, $P(A \cap B) = P(A) \cdot P(B)$, meaning that A and B are independent.

Compute $P(A \cap C)$: For both dice to show the same number and the sum to be 7, there is no possible outcome. Therefore,

$$P(A \cap C) = 0.$$

Compare $P(A \cap C)$ with $P(A) \cdot P(C)$:

$$P(A) \cdot P(C) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}.$$

Clearly, $P(A \cap C) = 0 \neq \frac{1}{36} = P(A) \cdot P(C)$.
Thus, A and C are not independent.

Question 6

One of ten keys you have opens a door. If you try the keys one after another, what is the probability that that the door is opened

i) on the first attempt?

ii) on the second attempt?

iii) on the 10th attempt?

Question 6

i) on the first attempt?

P(1st attempt) = (1 way to open)/(10 keys)

ii) on the second attempt?

P(2nd attempt)

We know the first attempt failed, (9/10)*(1/9)

iii) on the 10th attempt?

Same logic

$(9/10)(8/9)(7/8)\ (6/7)(5/6)(4/5)(3/4)(2/3)(1/2) = 1/10$

See you next week!