CIT 5920
Recitation 5

Shenao, Sam, Tiffany, Qianyue, Shutong

Penn Engineering

Overview for Today

- Logistics
- Last Recitation Continued
- Materials Review
- Sample Exam Review

Logistics

- Midterm 1 is on **Tuesday, Oct. 15 (5:15-6:45)**
  - **Towne 217**
- HW3 grades are released
- HW4 was just posted and is due two weeks after midterm (Mon, Oct 28 at 11:59PM).

Midterm 1

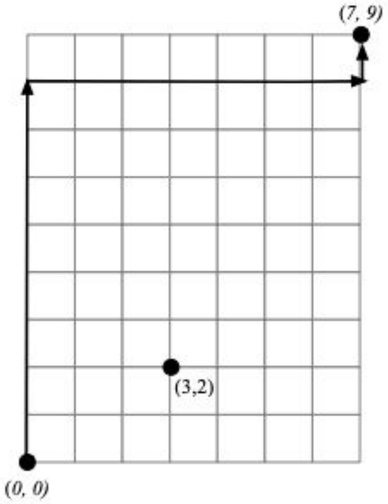
- Please take a look at announcement of Exam 1!
- Exam covers:
  - Sets, Relations, Functions, Counting
  - Basic Discrete Probability (bonus)
- You are allowed to bring **one double-sided** sheet of paper, either printed or handwritten.
- **No leaving early**
- Please provide **brief explanations** unless it is mentioned in the question.

Sample Exam - Exercise 5

Exercise 5 – Counting Monotonic Paths in a Grid

Consider a grid that is 7 units wide and 9 units tall. A *monotonic path* is one that begins at point $(0, 0)$ (the bottom left corner) and traverses to $(7, 9)$ (the top right corner), using only upward ($\uparrow$) and rightward ($\rightarrow$) moves. For example, the path on the right is a monotonic path:

$$(\uparrow, \uparrow, \uparrow, \uparrow, \uparrow, \uparrow, \uparrow, \uparrow, \rightarrow, \rightarrow, \rightarrow, \rightarrow, \rightarrow, \rightarrow, \rightarrow, \uparrow).$$



A. How many different monotonic paths are possible

B. How many such monotonic paths go through the point (3, 2)

Sample Exam - Solution 5a

A. How many different monotonic paths are possible

Solution: To get from point $(0, 0)$ to $(7, 9)$, one must move right 7 times and up 9 times, regardless of the order. This is equivalent to asking in how many ways we can arrange 7 rightward moves and 9 upward moves.

The number of ways to choose k items from n items (without regard to order) is given by the binomial coefficient:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

In this case, we want to choose 7 rightward moves from a total of $7 + 9 = 16$ moves. So, $n = 16$ and $k = 7$:

$$\binom{7+9}{7} = \binom{16}{7} = \frac{16!}{7!9!}$$

This gives the number of monotonic paths from $(0, 0)$ to $(7, 9)$.

Sample Exam - Solution 5b

B. How many such monotonic paths go through the point (3, 2)

Solution: To solve this, we break the problem into two parts:

1. The number of paths from $(0,0)$ to $(3,2)$.
2. The number of paths from $(3,2)$ to $(7,9)$.

For the first part, we need 3 rightward moves and 2 upward moves, a total of 5 moves. This is equivalent to choosing 3 rightward moves from 5 moves:

$$\binom{3+2}{3} = \binom{5}{3} = \frac{5!}{3!2!}$$

For the second part, from $(3,2)$ to $(7,9)$, we need 4 rightward moves and 7 upward moves, a total of 11 moves. This is equivalent to choosing 4 rightward moves from 11 moves:

$$\binom{4+7}{4} = \binom{11}{4} = \frac{11!}{4!7!}$$

The total number of paths that go through $(3,2)$ is the product of these two values.

Sample Exam - Exercise 6

A. There are three different fields for the students to practice on. How many different ways are there to assign the 27 players to the 3 fields in teams of 9?

$$\text{Solution: } \binom{27}{9} \times \binom{18}{9} \times \binom{9}{9}$$

B. How many ways are there to assign the 27 players to 3 teams of 9, without regard for which team is on which field?

$$\text{Solution: } \frac{\binom{27}{9} \times \binom{18}{9}}{3!} \times \binom{9}{9}$$

Sample Exam - Exercise 6 Continued

C. How many ways are there to assign the 27 players to 3 teams of 9, and for each team to choose 1 of its players as captain?

$$\text{Solution: } \frac{\binom{27}{9} \times \binom{18}{9}}{3!} \times 9^3$$

D. One of the teams plays 10 games against teams from other schools, ending the season with a 7-3 record (they won 7 games out of the 10 games they played, and lost 3 games). How many different sequences of wins and losses could have led to this outcome?

$$\text{Solution: } \binom{10}{7} = \frac{10!}{7!3!} = \binom{10}{3}$$

Sample Exam - Exercise 7a

Exercise 7 – Counting Solutions

How many non-negative ($x_i \geqslant 0$) integer solutions exist for the following systems of equations or inequations (remember that a system means that all of these need to be satisfied simultaneously)?

A.

$$x_1 + x_2 + x_3 + x_4 = 21$$

Sample Exam - Solution 7a

Solution: To solve this, we can use the "stars and bars" method. Imagine we have 21 stars in a row and we want to divide them into 4 groups using $4 - 1 = 3$ bars. Each group represents the value of one of the variables x_1, x_2, x_3, and x_4. The number of stars in each group gives the value of the corresponding variable.

For example, the configuration:

$$* * * * * | * * * * | * * * * * * | * * * * *$$

represents the solution $x_1 = 5$, $x_2 = 4$, $x_3 = 6$, and $x_4 = 6$.

The problem then reduces to determining the number of ways to arrange 21 stars and 3 bars. This is equivalent to choosing 3 positions out of 24 for the bars, which is:

$$\binom{21 + 4 - 1}{4 - 1} = \binom{24}{3}.$$

Sample Exam - Exercise 7b

$$
\begin{aligned}
x_1 + x_2 + x_3 + x_4 &\leqslant 21 \\
x_2 &\geqslant 2 \\
x_3 &\geqslant 3 \\
x_4 &\leqslant 19
\end{aligned}
$$

Sample Exam - Solution 7b

Solution: Using the "stars and bars" concept, we first introduce a slack variable x_5 to account for the inequality:

$$x_1 + x_2 + x_3 + x_4 + x_5 = 21$$

with the constraint $x_5 \geqslant 0$.

Given the constraints $x_2 \geqslant 2$ and $x_3 \geqslant 3$, we allocate 2 and 3 units respectively to x_2 and x_3. This modifies the equation to:

$$x_1 + x_2 + x_3 + x_4 + x_5 = 16$$

The constraint $x_4 \leqslant 19$ is automatically satisfied as x_4 cannot exceed 19 in any valid solution.

Now, we need to find non-negative integer solutions for:

$$x_1 + x_2 + x_3 + x_4 + x_5 = 16$$

Using "stars and bars", this is equivalent to arranging 16 stars with 4 bars, which gives:

$$\binom{20}{4}.$$

Sample Exam - Exercise 7c

c.

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &\leqslant 21 \\ x_2 + x_3 &= 5 \end{aligned}$$

Sample Exam - Solution 7c

Solution: Using the "stars and bars" concept, we first introduce a slack variable x_5 to account for the inequality:

$$x_1 + x_2 + x_3 + x_4 + x_5 = 21$$

with the constraint $x_5 \geq 0$.

Given the constraint

$$x_2 + x_3 = 5, \tag{1}$$

we can rewrite the equation as:

$$x_1 + x_4 + x_5 = 16 \tag{2}$$

Now, we need to find non-negative integer solutions to this equation.

Using "stars and bars", this is equivalent to arranging 16 stars with 2 bars, which gives:

$$\binom{18}{2}$$

Finally we combine the number of ways to fulfill the constraint in (1) with the number of ways to fulfill the constraint in (2), using the product rule, to get the total number of solutions to the combined system of equations:

$$\binom{6}{1} \times \binom{18}{2}.$$

Sample Exam - Exercise 8

Exercise 8 – Probability

There are 15 oranges that are being thrown randomly into 6 baskets. Each basket has enough capacity to hold all 15 oranges. What is the probability that the oranges don't all fall in the same basket? For the purposes of this problem, think of the oranges as distinct and the baskets as distinct.

Sample Exam - Solution 8

Solution: To solve this problem, we'll first determine the total number of ways the oranges can be thrown into the baskets, and then we'll determine the number of ways in which all the oranges can fall into the same basket. We will be using the *complement of an event*: The probability that the oranges don't all fall into the same basket will be 1 minus the probability that they all fall into the same basket.

- **Total number of ways the oranges can be thrown into the baskets:** Since there are 6 baskets and each orange can be thrown into any of the 6 baskets, the total number of ways the oranges can be thrown is 6^{15}.
- **Number of ways all the oranges fall into the same basket:** There are 6 baskets, and if all the oranges fall into one basket, there are 6 ways this can happen (one for each basket).
- **Probability that all the oranges fall into the same basket:**

$$P(\text{all in one basket}) = \frac{\text{number of ways all oranges fall into one basket}}{\text{total number of ways the oranges can be thrown}}$$

$$P(\text{all in one basket}) = \frac{6}{6^{15}}$$

- **Probability that the oranges don't all fall into the same basket:**

$$P(\text{not all in one basket}) = 1 - P(\text{all in one basket}$$

$$P(\text{not all in one basket}) = 1 - \frac{6}{6^{15}}.$$

The complement of an event in probability theory represents all outcomes that are not in the event itself. Utilizing the complement is a powerful tool, especially when it's challenging or cumbersome to directly compute the probability of a particular event. By calculating the probability of the complement (which might be simpler) and subtracting it from 1, we can indirectly find the desired probability. In the context of the oranges and baskets problem, while determining the probability of all oranges falling into one specific basket might be straightforward, calculating the probability for various other combinations can be complex. Hence, it's easier to compute the probability of the complement event "all in one basket" and subtract it from 1 to get the probability of the event "not all in one basket."

Question 1

Exercise 4

How many solutions are there to this $x_1 + x_2 + x_3 \leq 12$ but under each of the following constraints:

A. Each of the x_i must be strictly positive integers.

Answer 1a

Solution: This is a stars and bars problem.
First we add another variable x_4, and this variable will take whatever is left after x_1, x_2, and x_3 have their numbers chosen. So instead of finding solutions to $x_1 + x_2 + x_3 \leqslant 12$, we find those to $x_1 + x_2 + x_3 + x_4 = 12$.

For this questions x_1, x_2, and x_3 all need to be positive (cannot equal to zero), so we need to further manipulate this equation. We can take out 1 and assign/give it to each variable except x_4 to make sure x_1, x_2, and x_3 will at least have 1 to remain positive. We will denote this by replacing x_1, x_2, and x_3 with y_1, y_2, and y_3 where $y_1 = x_1 - 1$, $y_2 = x_2 - 1$, and $y_3 = x_3 - 1$.

Now the equation becomes: $y_1 + y_2 + y_3 + x_4 = 9$, where all variables are non-negative. A typical stars-and-bars problem. Now we applies stars and bars. There are 9 integers and 3 separators, and we choose the positions for those 3 separators which is $\binom{9+3}{3} = \binom{12}{3}$.

Question 1b

How many solutions are there to this inequality $x_1 + x_2 + x_3 \leqslant 12$, but under each of the following constraints:

B. Each of the x_i must be strictly positive integers, and $x_1 + x_2 \geqslant 5$.

Answer 1b

Solution: First, from part A, we already know the total without the last constraint that $x_1 + x_2 \geqslant 5$. First, since x_3 must be positive, let $y_3 = x_3 - 1$ to get $x_1 + x_2 + y_3 + x_4 = 11$. We will now case on the possible values of x_1.

- If $x_1 = 1$, then we have $x_2 + y_3 + x_4 = 10$ where $x_2 \geqslant 4$. Letting $y_2 = x_2 - 4$, this is the same as $y_2 + y_3 + x_4 = 6$.
- If $x_1 = 2$, then we have $x_2 + y_3 + x_4 = 9$, where $x_2 \geqslant 3$. Again, using $y_2 = x_2 - 3$, this translates into $y_2 + y_3 + x_4 = 6$.
- Notice for all $1 \leqslant x_1 \leqslant 4$, after transformations we have to find the number of ways to solve $y_2 + y_3 + x_4 = 6$ for non-negative y_is (and x_4). There are 6 stars and 2 bars, so this gives $\binom{8}{2}$ ways to solve this equation. For all possible 4 values of x_1, this is $4\binom{8}{2}$.
- If $x_1 \geqslant 5$, then we have $x_1 + x_2 + y_3 + x_4 = 11$, where $x_1 \geqslant 5$ and $x_2 \geqslant 1$. We can again give 5 to x_1 ($y_1 = x_1 - 5$) and 1 to x_2 ($y_2 = x_2 - 1$) to get $y_1 + y_2 + y_3 + x_4 = 5$, where all the y_is (and x_4) are non-negative. Here, we have 5 stars and 3 bars, so we get $\binom{8}{3}$ ways to solve this equation.

Thus adding up all the cases, we see the total number is $\binom{8}{3} + 4\binom{8}{2}$.

Question 1c

How many solutions are there to this inequality $x_1 + x_2 + x_3 \leqslant 12$, but under each of the following constraints:

C. Each of the x_i must be strictly positive integers, and $x_3 < 5$.

Answer 1c

Solution: First, from part A, we already know the total without the last constraint that $x_3 < 5$. Setting a strict upper bound on a variable can be difficult in this setting, but setting a lower one can be easier. Thus we can find all solutions in which $x_3 \geqslant 5$, and subtract this from the value found in A.

We will do this by the same method as in A. We first need to give 1 to each of x_1 and x_2 to ensure they are positive. However, here we also need to give 5 to x_3 to ensure it is at least 5. Thus we can set $y_1 = x_1 - 1$, $y_2 = x_2 - 1$, and $y_3 = x_3 - 5$ to get $y_1 + y_2 + y_3 + x_4 = 5$, where all the variables here are non-negative.

Again, we can apply the stars and bars formula. There are 5 integers and 3 separators, and we choose the positions for the 3 separators which is $\binom{5+3}{3} = \binom{8}{3}$.

Remembering that we wanted the opposite of this though, we need to subtract from A to get $\binom{12}{3} - \binom{8}{3}$.

See you next week!

Good luck on the exam!!!

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