Expectation practice

Linearity of expectation

PollEv.com/bhusnur4

Indicator random variables

Questions that are easier with indicators

Expectation of a random variable

$$E[X] = \sum_{\substack{i \text{ ranges} \\ \text{over all values} \\ \text{taken by } X}} i \, P(X = i)$$

average value / mean of X

Q. bit.ly/CIT5921024 1

X - amount of money won by my ticket

values for X

if win - 100,000,000

if lose - 0

$$E[X] = 10^8 * P(\text{winning}) + 0 * P(\text{losing}) \quad - \text{①}$$

$$= 10^8 * \frac{1}{(50)^6}$$

$$|\text{Sample space}| = \underline{50} * \underline{50} * \underline{50} * \underline{50} * \underline{50} * \underline{50}$$

$$= (50)^6$$

$$\frac{\#\text{ of ways our event occurs}}{|\text{Sample space}|}$$

Q. bit.ly/C1759210242

7 cows 3 sheep 2 animals picked

$E[C]$ where C is # of cows picked

$$C = \sum_{i=0}^{2} P(C=i) * i$$

$$= 0 * P(C=0) + 1 * P(C=1) + 2 * P(C=2)$$ (2)

$P(C=1)$ = prob of picking 1 cow

$$= \frac{7}{10} * \frac{3}{9} + \frac{3}{10} * \frac{7}{9}$$

$$\frac{\overset{\text{cow}}{7} * \overset{\text{sheep}}{3}}{\binom{10}{2}}$$ ← # of ways to pick 2 animals

$$P(C=2) = \frac{\binom{7}{2}}{\binom{10}{2}}$$

$\binom{7}{2}$ ← pick 2 cows from 7

$\binom{10}{2}$ ← pick any 2 animals.

Linearity of Expectation

$$E[X+Y] = E[X] + E[Y]$$

X and Y do not have to be independent

Q. Expected value of the sum of rolling 2 dice?

$$E[D_1] = \frac{1}{6}(1+2+3+\dots+6)$$

$$E[D_2] = \frac{1}{6}(1+2+3+\dots+6)$$

$$E[D_1 + D_2] = E[D_1] + E[D_2]$$
$$= 3.5 + 3.5 = 7$$

$2 * P(\text{sum of 2 dice is 2})$

$$\frac{1}{36}$$

C_1 – r.v. that takes value 1 if cow 1 is picked and 0 otherwise

C_2 – r.v. that takes value 1 if cow 2 is picked

⋮

$$C = C_1 + C_2 + \dots + C_7$$

Linearity

$$E[aX + bY] = aE[X] + bE[Y]$$

(2)

a, b are some real nos

Note: $E[XY] \overset{??}{=} E[X]E[Y]$ not always true

Indicator random variable

A random variable that takes value

$\underline{1}$ when some event A occurs

0 when it does not

Common notation I_A – indicator r.v. corresponding to A

Q. 5 card hand dealt from a shuffled deck of cards. What is the expected number of hearts?

H – r.v. corresponding to # of hearts.

values of H possible are $0, 1, 2, 3, 4, 5$.

$$E[H] = 0 * P(0 \text{ hearts}) + 1 * P(1 \text{ heart}) + \dots + 5 * P(5 \text{ hearts})$$

$$\frac{\binom{13}{5}}{\binom{52}{5}}$$

This should be $\frac{5}{4}$

$\underline{\text{Goal}}$: want to write H as sum of random vars.

H_1 r.v. that is $\underline{1}$ if 1^{st} card is a heart and 0 otherwise

indicator of 1^{st} card being heart.

H_2, H_3, H_4, H_5 defined similarly

$$H = H_1 + H_2 + H_3 + H_4 + H_5$$

$$E[H] = E[H_1 + H_2 + \dots + H_5]$$

which by linearity

$$= E[H_1] + E[H_2] + \dots + E[H_5]$$

$$E[H_1] = 1 \times P(1^{st} \text{ card being heart}) + 0 \times \underline{\qquad}$$

$$= P(1^{st} \text{ card being heart}) = \frac{1}{4}$$

<u>Result</u> $\quad E[I_A] = P(A \text{ occuring})$

$$E[H_2] = P(2^{nd} \text{ card being heart})$$

$=$

1st card not being heart

OR

1st card being heart

$$\frac{39}{52} \times \frac{13}{51} + \frac{13}{52} \times \frac{12}{51}$$

$$\frac{39 \times 13 + 13 \times 12}{52 \times 51} = \frac{13\,\cancel{(39+12)}}{52 \times \cancel{51}}$$

Each of H_i has $E[H_i] = \frac{13}{52} = \frac{1}{4}$

$$\therefore E[H] = E[H_1] + E[H_2] + \dots + E[H_5]$$

$$= \frac{5}{4}$$

Derangements / Hat check problem / matching problem

n people
n hats corresponding to the people
each hat belongs to 1 and only 1 person

n people get drunk & then pick up one random hat each.

What's the expected # of people who get their hat back?

X - # of people who get hat back.

$$\begin{array}{cccccc} P_1 & P_2 & P_3 & P_4 & \cdots & P_n \\ H_1 & H_2 & H_3 & H_4 & \cdots & H_n \end{array} \Bigg\} \text{ pre drunk}$$

$$\begin{array}{cccccc} P_1 & P_2 & P_3 & P_4 & \cdots & P_n \\ H_1 & H_5 & H_6 & H_4 & \cdots & H_2 \end{array} \Bigg\} \text{ possible picked up hats}$$

range of values for X = ??

$$E[X] = \sum_{i=0} i \; P(i \text{ people got their hat back})$$

let X_1 - indicator of person 1 getting their hat back. 1 if P_1 gets H_1. 0 otherwise

X_2 - 1 if P_2 gets H_2. 0 otherwise

⋮

X_n - 1 if P_n gets H_n. 0 otherwise

$$X = X_1 + X_2 + \ldots + X_n \quad (A)$$

Apply linearity of expectation to (A)

$$E[X] = E[X_1] + E[X_2] + \dots + E[X_n] \quad \text{B}$$

$$E[X_1] = 1 * P(P_1 \text{ gets } H_1) + 0 * \_\_\_\_$$

$$= P(1^{st} \text{ person got their hat back})$$

$$= \frac{1}{n}$$

$$E[X_2] = P(2^{nd} \text{ person got their hat back})$$

$$= \frac{1}{n}$$

Plug this back into B

$$E[X] = 1/n + 1/n + \dots + 1/n = 1$$