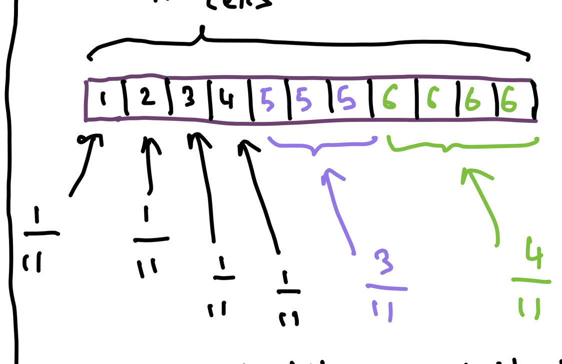
Lecture 11: Rules of Probability

First, let's figure out the probability distribution given that:
- 6 is 4-times more likely
- 5 is 3-times more likely

11 cells



Number of cells – one for each number
- 3 for 5
- 4 for 6

So the probability distribution is:

$$\begin{cases} \mathbb{P}[1] = \mathbb{P}[2] = \mathbb{P}[3] = \mathbb{P}[4] = \frac{1}{11} \\ \mathbb{P}[5] = \frac{3}{11} \\ \mathbb{P}[6] = \frac{4}{11} \end{cases}$$

1. RULES OF PROBABILITY

1.1 ADDITION RULE

Recall an event is a subset of the sample space

- FOR MUTUALLY EXCLUSIVE EVENTS (only one or the other occurs, but never both)

  Events A and B are mutually exclusive if

  $A \cap B = \emptyset$

  then

  $$\mathbb{P}[A \cup B] = \mathbb{P}[A] + \mathbb{P}[B]$$

  (UNION of events = either occurs)

- GENERAL ADDITION RULE

  $$\mathbb{P}[A \cup B] = \mathbb{P}[A] + \mathbb{P}[B] - \mathbb{P}[A \cap B]$$

  (INTERSECTION of events = both occur)

EX1. When rolling a die, $S = \{1, 2, 3, 4, 5, 6\}$

the event Even $= \{2, 4, 6\}$ and Odd $= \{1, 3, 5\}$ are mutually exclusive since Even $\cap$ Odd $= \emptyset$

therefore

$$\mathbb{P}[\text{Odd} \cup \text{Even}] = \underbrace{\mathbb{P}[\text{Odd}]}_{3/6} + \underbrace{\mathbb{P}[\text{Even}]}_{3/6}$$

EX2. What is the probability/likelihood of drawing a card that is a King or a Heart from a 52-card deck?

$\mathbb{P}[\text{King}] = \frac{4}{52}$ ← 4 kings in the deck; 52 total cards in the deck

$\mathbb{P}[\heartsuit] = \frac{13}{52}$ ← 13 cards are ♥ hearts

$$\mathbb{P}[\text{King} \cup \heartsuit] = \mathbb{P}[\text{King}] + \mathbb{P}[\heartsuit] - \mathbb{P}[\text{King} \cap \heartsuit]$$

$$= 4/52 + 13/52 - 1/52$$

$$= \boxed{16/52}$$

there is only one event/card in this INTERSECTION King of Hearts

1.2 COMPLEMENT RULE

(my favorite)

The complement of an event A, denoted $\bar{A}$ or A^c, consists of ALL outcomes NOT in A.

read as BAR A or A BAR (or in some situations NOT A)

$$\mathbb{P}[\bar{A}] = 1 - \mathbb{P}[A]$$

EX1. What is the probability of not rolling a 6 on a fair die?

$$\mathbb{P}[\text{Six}] = \frac{1}{6} \qquad \mathbb{P}[\overline{\text{Six}}] = 1 - \frac{1}{6} = \frac{5}{6}$$

1.3 INCLUSION-EXCLUSION PRINCIPLE

Generalization to the union of more than two events:

$$\mathbb{P}[A \cup B \cup C] = \mathbb{P}[A] + \mathbb{P}[B] + \mathbb{P}[C] - \mathbb{P}[A \cap B] - \mathbb{P}[B \cap C] - \mathbb{P}[A \cap C] + \mathbb{P}[A \cap B \cap C]$$

2. CONDITIONAL PROBABILITY

The conditional probability is the probability that event A will happen given that event B has occurred.

$$\mathbb{P}[A \mid B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]} \quad \text{with } \mathbb{P}[B] > 0$$

probability of A "knowing" B (or "assuming" B)

EX1. From a 52-card deck, what is the probability that a card that we pick is a Queen given it is a face card? (means it is Jack, Queen or King)

$$\mathbb{P}[\text{Queen} \mid \text{Face Card}] = \frac{\mathbb{P}[\text{Queen} \cap \text{Face Card}]}{\mathbb{P}[\text{Face Card}]}$$

Total cards: 52

Total face cards: 12 (Jacks, Queen, King)

Total Queens: 4

$$\mathbb{P}[\text{Queen} \mid \text{Face Card}] = \frac{\overbrace{\mathbb{P}[\text{Queen} \cap \text{Face Card}]}^{4/52}}{\underbrace{\mathbb{P}[\text{Face Card}]}_{12/52}} = \frac{\frac{4}{52}}{\frac{12}{52}}$$

$$= \frac{4}{\cancel{52}} \times \frac{\cancel{52}}{12} = \frac{4}{12} \left[= \frac{1}{3}\right]$$

EX2. Rolling dice with conditions

Two dice are rolled. What is the probability that the sum is 8 given the first die shows as 5?

- Sample space for a two-die experiment would usually be

$$\{1,2,3,4,5,6\} \times \{1,2,3,4,5,6\} = \begin{Bmatrix} (1,1), & (1,2), & (1,3) \ldots & (1,6), \\ (2,1), & (2,2) & \ldots & (2,6), \\ \vdots & \vdots & & \vdots \\ (6,1) & \ldots & \ldots & (6,6) \end{Bmatrix}$$

- But here given we know the first die is 5 we are restricted (like in EX1 with Face Cards) to the sample space where the first die is 5:

$$S = \{(5,1), (5,2), (5,3), (5,4), (5,5), (5,6)\}$$

- Given that, the only outcome that adds up to 8 is (5,3) so the event only has one outcome

- $\mathbb{P}[\text{Sum is } 8 \mid \text{First die is } 5] = \frac{1}{6}$ ← (5,3); $|S| = 6$

Next Thursday: Independence + Random Variables