CIT 5920
Recitation 4

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Overview for Today

- Logistics
- Question 1 (10 minutes)
- Question 2 (10 minutes)
- Question 3 (3 minutes)
- Question 4 (20 minutes)
- Question 5 (5 minutes)
- Question 6 (10 minutes)
- Question 7 (10 minutes)
- Question 8 (20 minutes)

Logistics

- HW1 grades will be released this weekend
- HW2 grades will be released soon
- HW3 is due next Monday

Review

Permutation: A permutation is an arrangement of distinct objects in a specific order. Order matters in permutations.

$$P(n,r) = \frac{n!}{(n-r)!}$$

Review

Combination: A combination is a selection of objects without considering the order. Order does not matter in combinations.

$$C(n,r) = \frac{n!}{r!(n-r)!}$$

Review

Example:

Imagine a race with 3 runners: Alice, Bob, and Carol. How many different ways can they finish in 1st, 2nd, and 3rd place?

Review

- Alice, Bob, Carol (ABC)
- Alice, Carol, Bob (ACB)
- Bob, Alice, Carol (BAC)
- Bob, Carol, Alice (BCA)
- Carol, Alice, Bob (CAB)
- Carol, Bob, Alice (CBA)

→ 3C3 = 1

3P3 = 6 or **Product rule**: 3x2x1 = 6

What if we only care about how many different ways can we choose a set of 3 runners to finish, regardless of who finishes 1st, 2nd, or 3rd?

Order does not matter!

Review

Bijection Explanation with Racing Medals:

Set 1: All possible ways to award medals (Gold, Silver, Bronze).

Set 2: All possible ways to assign 1st, 2nd, and 3rd place to runners.

For each possible way to assign the medals (permutation of 3 objects), there is exactly one way to assign the runners to positions.

Review

Stars and bars

If you want to distribute k indistinguishable objects into n distinguishable bins (groups), the number of ways to do so is given by:

$$C(k+n-1, n-1)$$

This is the number of ways to place n-1 dividers (bars) between k objects (stars).

Review

How many non negative solutions for this equation

$$x_1 + x_2 + x_3 = 4$$

★ | ★ | ★ ★ $\longrightarrow$ $x_1 = 1, x_2 = 1, x_3 = 2$

Review

k = 4 (stars)

n = 3 (bins)

$$\binom{k+n-1}{n-1} = \binom{6}{2}$$

Question 1

Exercise 1

In how many ways can we form 4 letter words using 4 letters from the word “ADDRESS”? To clarify, assume each of those letters was a block (think about Scrabble or Words with Friends if you have played those games). This question says you have only 7 letter blocks and you can only use those blocks. A word like “ADRE” is possible but a word like “ADRA” is not possible because you have only 1 “A”.

Answer 1

A D D R E S S

no dups

1 dups

2 dups

A D R E S

(D D) or (S S) + (left over)

D D S S

Pick 4 of these

Pick 1 of these

Pick 2 of these

Arrange them!

Arrange them!

Arrange them!

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Answer 1 (cont.)

Solution:

$$P(5,4)+\binom{2}{1}\binom{4}{2}\frac{4!}{2!}+\binom{2}{2}\binom{4}{0}\frac{4!}{2!2!}$$

There are 5 letters; A, R, E each appear 1 time. D and S each appear 2 times.
This can be considered in 3 cases. When all letters are distinct, there are $P(5,4) = 5!$ ways to order them. In the case that one letter (either D or S) is repeated twice, there are $\binom{2}{1}$ ways to pick one of them and $\binom{4}{2}$ ways to pick 2 from the remaining 4 letters. You can arrange these letters in $4!$ ways; since one letter is a duplicate, you divide by $2!$. Similarly, in the case where two letters are repeated (D and S are both used), there's only 1 way to choose these letters (i.e. you choose two letters from the 2 duplicate letters and no letters from the remaining letters... $\binom{2}{2}\binom{4}{0} = 1$). Again, there are $4!$ ways to arrange these letters; since there are 2 duplicates, you would divide the result by $2!2!$ to address them.

Question 2

Exercise 2

There are 32 first year MCITs and as part of 5910 recitation they are told to randomly pair up and do pair programming. How many different pairings are possible? Note that we are counting the distinct pairings and not the number of ways to pick one pair.

For example, with 4 people - A, B, C, and D, we could have A with B or A with C or A with D, and in each case, the remaining two are in a pair. These are the possibilities, so the answer is 3.

Hint: The answer is not $\binom{32}{3}$.

Answer 2

Solution: First, let's work with a simpler question: in how many ways can you generate *only* the first pair of students (not the distinct "pairings of all 32" we ultimately need)?

It is $\binom{32}{2}$.

What about the second pair? Regardless of who was in the first, we now have only 30 students from whom to choose: $\binom{30}{2}$.

Continuing in this pattern, it is tempting to just say: $\prod_{m=1}^{16} \binom{2m}{2}$ students.

However, note that if Student A and Student B are in a pair, it doesn't matter whether it was the first pair, last pair or something else.

Intuition: we need to divide by the number of permutations among the sixteen pairs we select because their order does not matter–just that they contain two particular students: $\frac{1}{16!} \prod_{m=1}^{16} \binom{2m}{2}$.

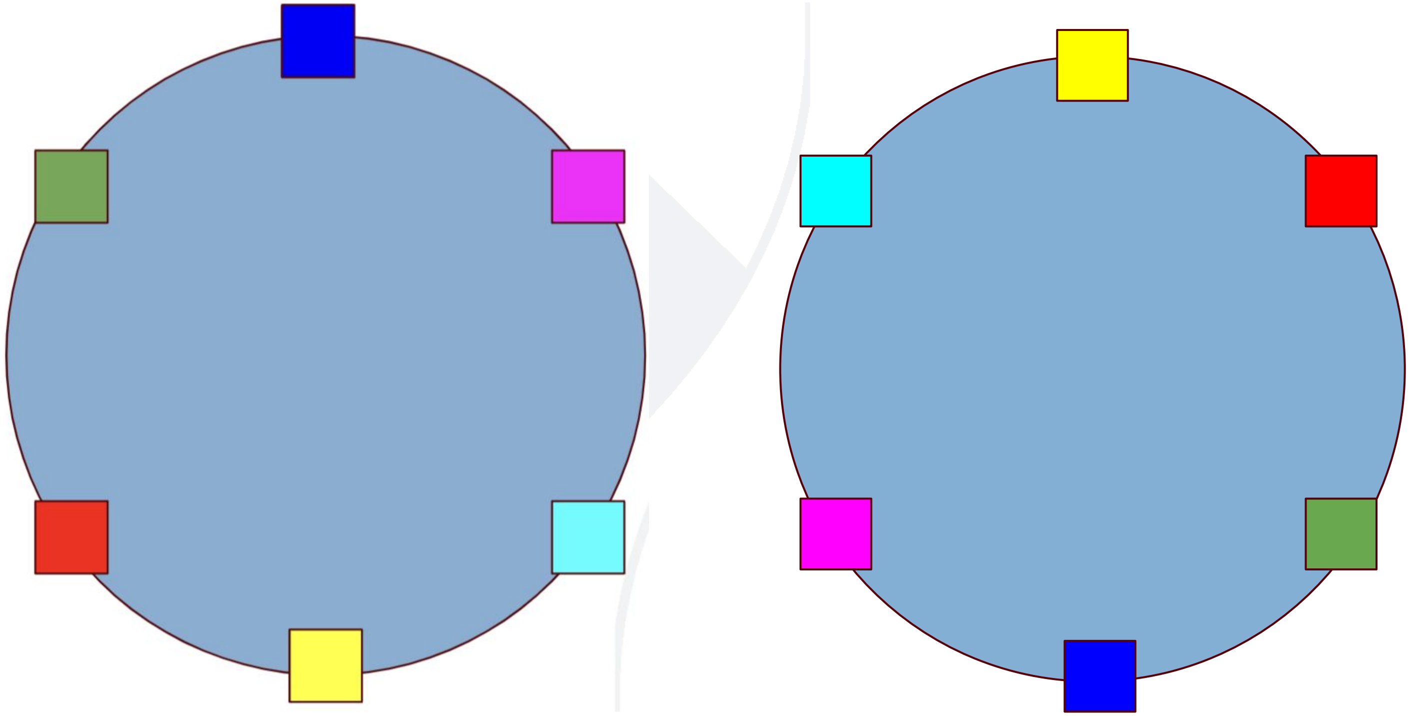
Expanding each of the multiplied terms above simplifies to: $\frac{32!}{16! \cdot 2^{16}}$.

Question 3

Exercise 3
In how many ways can 6 people be seated around a circular table? Be careful about not over-counting equivalent arrangements.

Answer 3

Solution: We can arrange the students in 6! ways if it was a line, but because of the circular symmetry from the table, any 6 of the similar rotations are the same seating. Thus we need to divide by 6 to get $\frac{6!}{6} = 5!$.



Question 4

Exercise 4

How many solutions are there to this $\boxed{x_1 + x_2 + x_3 \leq 12}$ but under each of the following constraints:

A. Each of the x_i must be strictly positive integers.

Answer 4a

Solution: This is a stars and bars problem.
First we add another variable x_4, and this variable will take whatever is left after x_1, x_2, and x_3 have their numbers chosen. So instead of finding solutions to $x_1 + x_2 + x_3 \leqslant 12$, we find those to $x_1 + x_2 + x_3 + x_4 = 12$.

For this questions x_1, x_2, and x_3 all need to be positive (cannot equal to zero), so we need to further manipulate this equation. We can take out 1 and assign/give it to each variable except x_4 to make sure x_1, x_2, and x_3 will at least have 1 to remain positive. We will denote this by replacing x_1, x_2, and x_3 with y_1, y_2, and y_3 where $y_1 = x_1 - 1$, $y_2 = x_2 - 1$, and $y_3 = x_3 - 1$.

Now the equation becomes: $y_1 + y_2 + y_3 + x_4 = 9$, where all variables are non-negative. A typical stars-and-bars problem. Now we applies stars and bars. There are 9 integers and 3 separators, and we choose the positions for those 3 separators which is $\binom{9+3}{3} = \binom{12}{3}$.

Question 4b

How many solutions are there to this inequality $x_1 + x_2 + x_3 \leqslant 12$, but under each of the following constraints:

B. Each of the x_i must be strictly positive integers, and $x_1 + x_2 \geqslant 5$.

Answer 4b

Solution: First, from part A, we already know the total without the last constraint that $x_1 + x_2 \geqslant 5$. First, since x_3 must be positive, let $y_3 = x_3 - 1$ to get $x_1 + x_2 + y_3 + x_4 = 11$. We will now case on the possible values of x_1.

- If $x_1 = 1$, then we have $x_2 + y_3 + x_4 = 10$ where $x_2 \geqslant 4$. Letting $y_2 = x_2 - 4$, this is the same as $y_2 + y_3 + x_4 = 6$.
- If $x_1 = 2$, then we have $x_2 + y_3 + x_4 = 9$, where $x_2 \geqslant 3$. Again, using $y_2 = x_2 - 3$, this translates into $y_2 + y_3 + x_4 = 6$.
- Notice for all $1 \leqslant x_1 \leqslant 4$, after transformations we have to find the number of ways to solve $y_2 + y_3 + x_4 = 6$ for non-negative y_is (and x_4). There are 6 stars and 2 bars, so this gives $\binom{8}{2}$ ways to solve this equation. For all possible 4 values of x_1, this is $4\binom{8}{2}$.
- If $x_1 \geqslant 5$, then we have $x_1 + x_2 + y_3 + x_4 = 11$, where $x_1 \geqslant 5$ and $x_2 \geqslant 1$. We can again give 5 to x_1 ($y_1 = x_1 - 5$) and 1 to x_2 ($y_2 = x_2 - 1$) to get $y_1 + y_2 + y_3 + x_4 = 5$, where all the y_is (and x_4) are non-negative. Here, we have 5 stars and 3 bars, so we get $\binom{8}{3}$ ways to solve this equation.

Thus adding up all the cases, we see the total number is $\binom{8}{3} + 4\binom{8}{2}$.

Question 4c

How many solutions are there to this inequality $x_1 + x_2 + x_3 \leqslant 12$, but under each of the following constraints:

C. Each of the x_i must be strictly positive integers, and $x_3 < 5$.

Answer 4c

Solution: First, from part A, we already know the total without the last constraint that $x_3 < 5$. Setting a strict upper bound on a variable can be difficult in this setting, but setting a lower one can be easier. Thus we can find all solutions in which $x_3 \geqslant 5$, and subtract this from the value found in A.

We will do this by the same method as in A. We first need to give 1 to each of x_1 and x_2 to ensure they are positive. However, here we also need to give 5 to x_3 to ensure it is at least 5. Thus we can set $y_1 = x_1 - 1$, $y_2 = x_2 - 1$, and $y_3 = x_3 - 5$ to get $y_1 + y_2 + y_3 + x_4 = 5$, where all the variables here are non-negative.

Again, we can apply the stars and bars formula. There are 5 integers and 3 separators, and we choose the positions for the 3 separators which is $\binom{5+3}{3} = \binom{8}{3}$.

Remembering that we wanted the opposite of this though, we need to subtract from A to get $\binom{12}{3} - \binom{8}{3}$.

Question 5

Exercise 5

How many 18 bit numbers exist that have exactly 8 0s and 10 1s if every zero must immediately be followed by a 1?

Answer 5

Solution: Every 0 must have a 1 after it, so we can break it into chunks. We want to arrange 01, 01, 01, 01, 01, 01, 01, 1, and 1. Basically, this is asking how many different places can we put in the left-over 2 1s. This is just stars and bars, where there are 8 bars (each 01) and 2 stars. Thus it becomes $\binom{2+8}{8} = \binom{10}{2}$.

_ 01 _ 01 _ 01 _ 01 _ 01 _ 01 _ 01 _ 01 _

1 1

Question 6

Exercise 6

Suppose a person rolls n identical 6 sided dice and they roll them all at once. How many distinct throws are possible? For clarity consider a scenario where there are 7 dice. Then we will consider 1122331 the same as 1112233. Also the same as 1123231.

Answer 6

Solution: We can do this with a clever application of stars and bars. After any roll, we can order the dice in increasing order (this doesn't matter what order since the dice are indistinguishable). Now imagine that we insert our 5 bars here. The dice to the left of the first bar are 1s, the dice between the first and second are 2s, and so on, until the dice to the right of the last bar are 6s. Each of these directly corresponds to a distinct throw. Thus we have n stars (each of the dice) and 5 bars. Therefore there are $\binom{n+5}{5}$ possible distinct throws.

Question 7

Exercise 7

In how many ways can 10 different rings be distributed among the fingers of one hand? It is assumed that any finger can hold all rings.

Answer 7

Solution: Step 1: Distribution of Rings to Fingers (Assuming Rings are Identical)
Initially, we treat the 10 rings as if they are identical, and we use the stars and bars method to determine how many ways we can distribute these identical rings among the 5 fingers. This step does not consider the distinctness of the rings; it only focuses on the number of rings on each finger.
Using the stars and bars method, the number of ways to distribute n identical items (rings) into r distinct bins (fingers) is given by:

$$\binom{n+r-1}{r-1}$$

$$\binom{10+5-1}{5-1} = \binom{14}{4} = \frac{14!}{4! \cdot 10!} = 1001 \text{ ways}$$

Step 2: Arranging the Distinct Rings on the Fingers
Now, we consider the distinctness of the rings. For each distribution obtained in step 1, we have 10! ways to arrange the 10 distinct rings. For example, if a finger is allocated one ring in a certain distribution, there are 10 different possibilities for which ring it could be, since we have 10 distinct rings.
Step 3: Total Number of Ways
To find the total number of ways to distribute the rings, we multiply the number of distributions (from stars and bars, where rings are considered identical) by the number of arrangements (permutations of distinct rings):

$$\text{Total Ways} = 1001 \text{ distributions} \times 10! \text{ arrangements}$$

$$\text{Total Ways} = 1001 \cdot 3,628,800$$

Question 8

Exercise 8

You are an environmental scientist working to protect a delicate ecosystem from pollution. You have 1000 samples of water from different parts of a river, and one is contaminated with a harmful chemical. Using 10 test strips, you need to identify the contaminated sample to prevent ecological damage and protect biodiversity. A single drop of contaminated water will turn the test strip positive permanently. You can put any number of drops on a test strip at once and you can reuse a test strip as many times as you'd like (as long as the results are negative). However, you can only run tests once per day and it takes seven days to return a result. How would you figure out the contaminated sample in as few days as possible?

Answer 8

Solution: Let's break it down a little bit.

Binary Basics: First, let's start by understanding how to represent numbers from 1 to 1000 using binary notation. How many bits are required? We can do this using 10 bits (for up to 1024 numbers). Why do we want to do this though? It is easier to uniquely identify a number using its binary representation.

Test Strip Mapping: If you had only one test strip and two samples, one of which is contaminated, how would you determine which sample is contaminated using the test strip? We could place a drop from the first sample on one half of the test strip and a drop from the second sample on the other half. The half that turns positive indicates the contaminated sample. Or, we could test just one, and if it wasn't positive, then the other must be.

Expanding the Scope: Now, imagine you have 4 samples. How can you use binary representation and two test strips to determine which sample is contaminated?

We can label the samples with binary numbers: 00, 01, 10, and 11. Now we use the first test strip for everything with a 1 in the first bit and the second test strip for everything with a 1 in the second bit. For example, if the first test strip comes back positive, then we know the first bit must be 1, but if it comes back negative, it must be 0. The combination of positive test strips will indicate exactly the contaminated sample.

Generalization: How can we generalize the method used for 4 samples to handle 1000 samples using 10 test strips?

We can label each sample with a unique 10-bit binary number. For each bit position, use a separate test strip on only those with a 1 in that position. The combination of positive test strips will give the unique binary representation of the contaminated sample.

Final Step: Given the results from the 10 test strips after 7 days, how can you identify the exact contaminated sample?

We can convert the combination of positive test strips back to its decimal representation to get the number of the contaminated sample.

Thus it can be done in 7 days.

See you next week!

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