CIT 5920
Recitation 3

Fall 2024

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Overview for Today

- Logistics
- Review
- Functions
  - Question 1 (3 minutes)
  - Question 2 (3 minutes)
  - Question 3 (3 minutes)
- Counting
  - Question 4 (5 minutes)
  - Question 5 (5 minutes)
  - Question 6 (5 minutes)
  - Question 7 (5 minutes)
  - Question 8 (5 minutes)
- Homework

Logistics

- HW2 is due Monday night (September 23rd) at 23:59 PM
- HW3 is being released September 23nd and is due next Monday, September 30th

Anti-symmetry

Definition: **if** the relation, R, contains **both** (a, b) and (b, a), **then** a = b must be true for the relation to be Anti-symmetric.

An alternative way to think of it is that a relation, R, is Anti-symmetric only if every time we have (a, b) in the relation we do not have (b, a) if a and b are not the same.

Anti-symmetric does not mean not Symmetric. They are essentially independent: relation can be both Symmetric and Anti-symmetric, just one of the two, or neither.

Symmetry vs. Anti-symmetry

Symmetry	Anti-symmetry
If: (a, b) is in R Then: (b, a) must also be in R	If: (a, b) is in R and (b, a) is in R Then: a = b Alternatively: If: (a, b) is in R and a ≠ b Then: (b, a) must not be in R

Anti-symmetry (cont.)

Not Symmetric, Not Anti-symmetric: **{(1, 2), (2, 1), (1, 3)}**

Symmetric, Not Anti-symmetric: **{(1, 2), (2, 1)}**

Not Symmetric, Anti-symmetric: **{(1, 3)}**

Symmetric, Anti-symmetric: **{(1, 1)}**

To check for Anti-symmetry we can use the same approach we use for testing Transitivity: if we can't provide a counter-example then we must accept that the relation is Anti-symmetric. In other words, if we can't show that it's not Anti-symmetric, then it must be Anti-symmetric.

Anti-symmetry (cont.)

Here's an example of an Anti-symmetric relation:

aRb if a <= b

The term (x, y) translates to x <= y

Let's say (x, y) is in the relation

When will (y, x) also be in the relation?

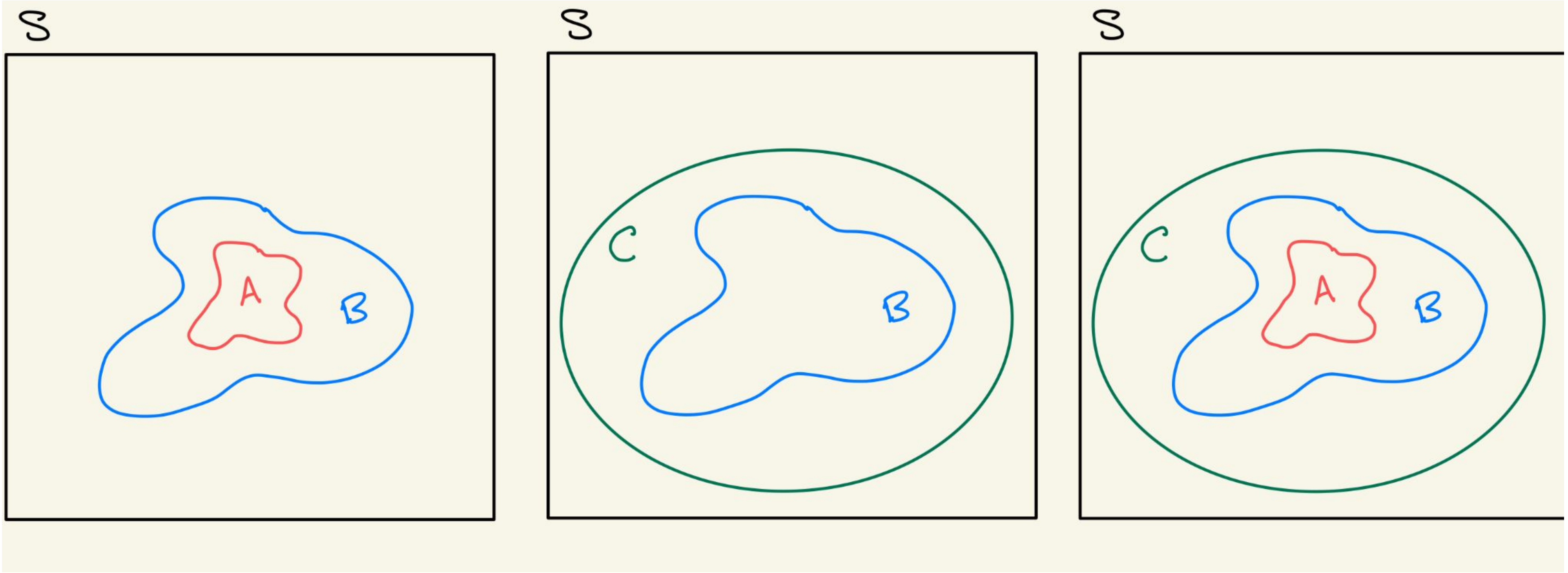
(y, x) translates to y <= x

The only time x <= y and y <= x can both be true is when:

x = y

Transitivity

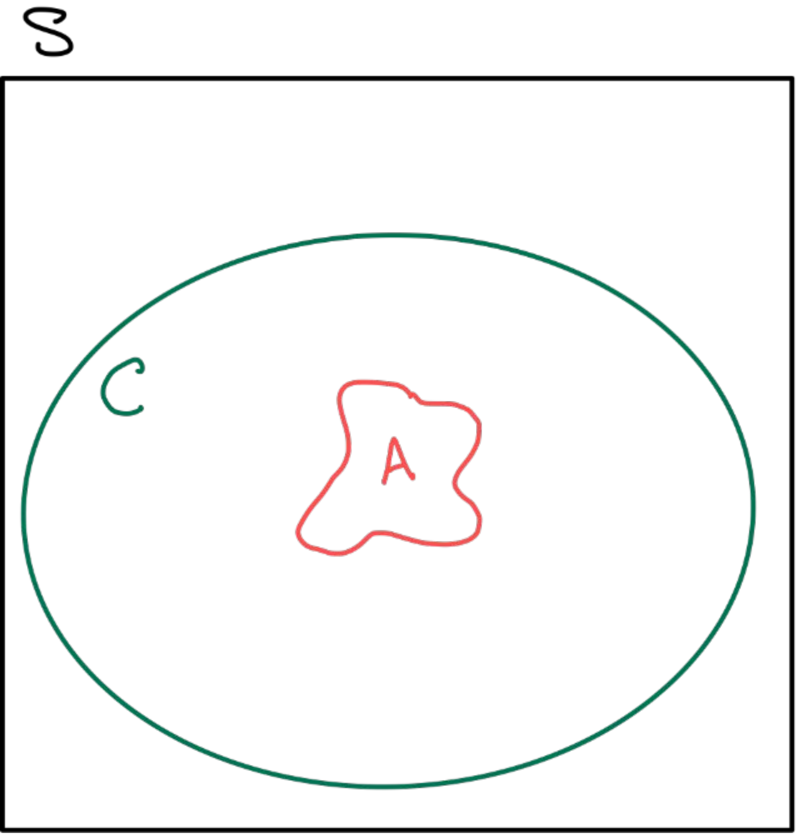
- Transitive: if whenever *aRb* and *bRc*, then *aRc*



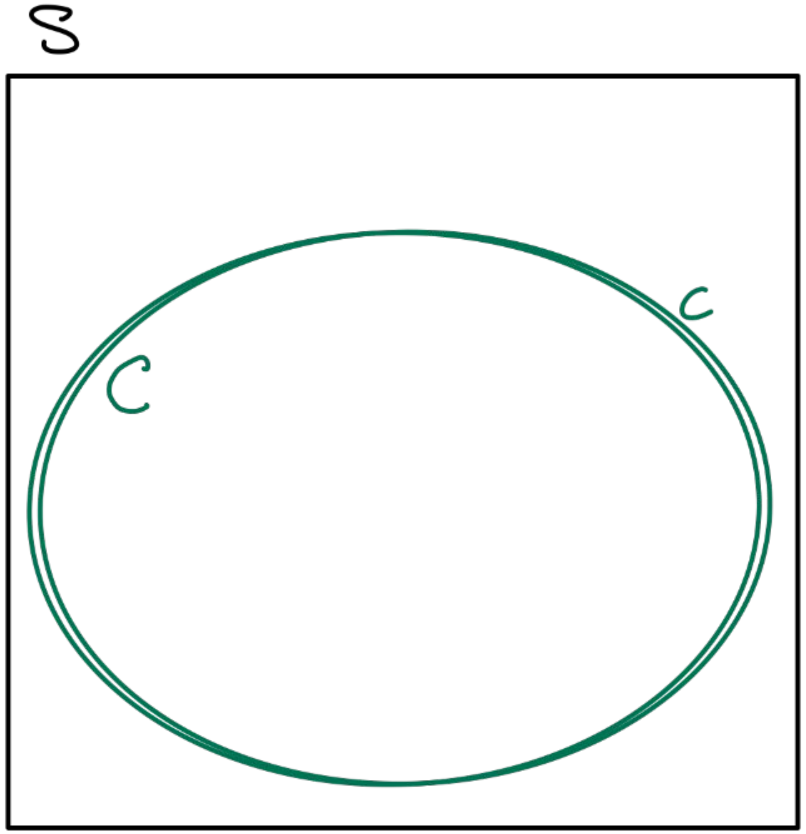
aRb bRc aRb and bRc

Transitivity

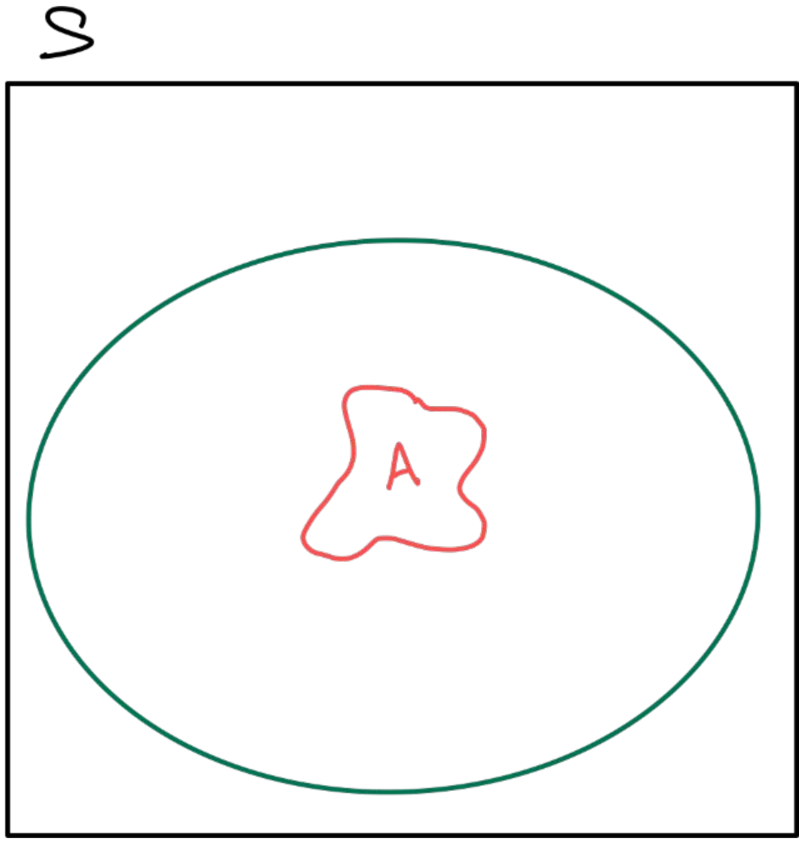
- Transitive: on S if whenever *aRb* and *bRc*, then *aRc*



aRc



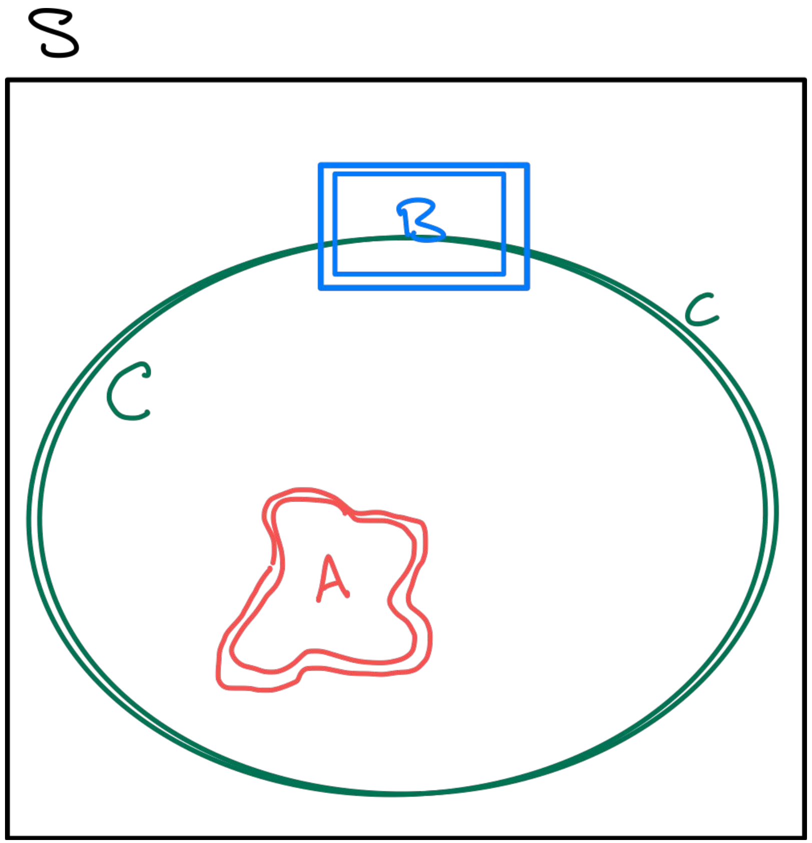
cRc



aRc and cRc

Reflexivity

- Reflexive: if **for all** elements *a* in set *S*, *aRa*.



aRa bRb cRc

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Review Question

Consider the set A={1,2,3} and the relation R on A defined as follows:

R={(1,1),(2,2),(3,3),(1,2),(2,3)}

Determine if the relation R is:

- Reflexive
- Symmetric
- Transitive
- Antisymmetric

Review Answer

Consider the set A={1,2,3} and the relation R on A defined as follows:

R={(1,1),(2,2),(3,3),(1,2),(2,3)}

Determine if the relation R is:

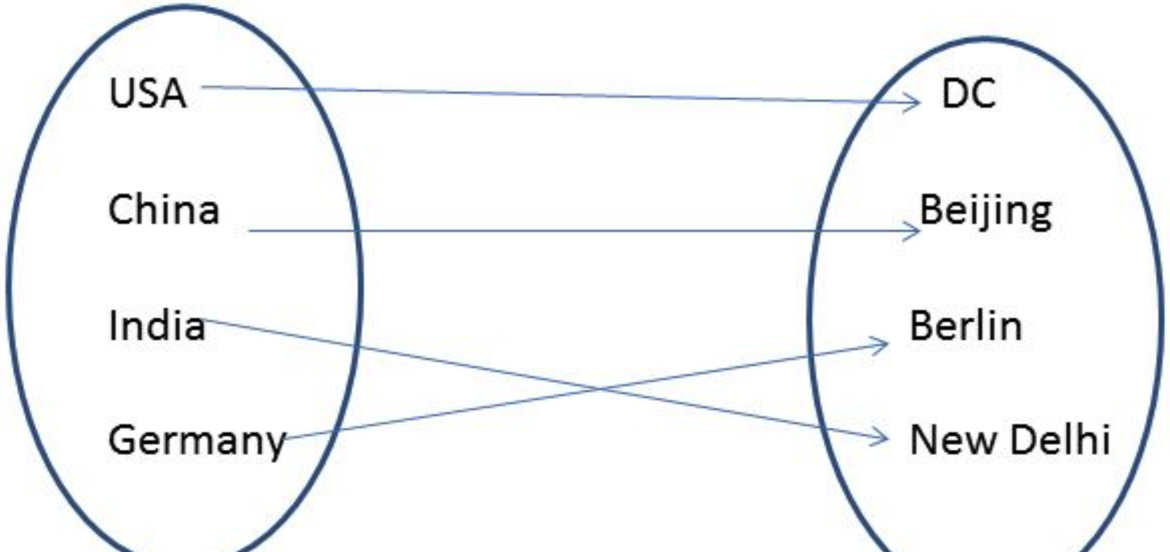
● Reflexive	Y
● Symmetric	N
● Transitive	N
● Antisymmetric	Y

Review

- Functions
- Injection (one-to-one)
- Surjections (onto)
- Bijections

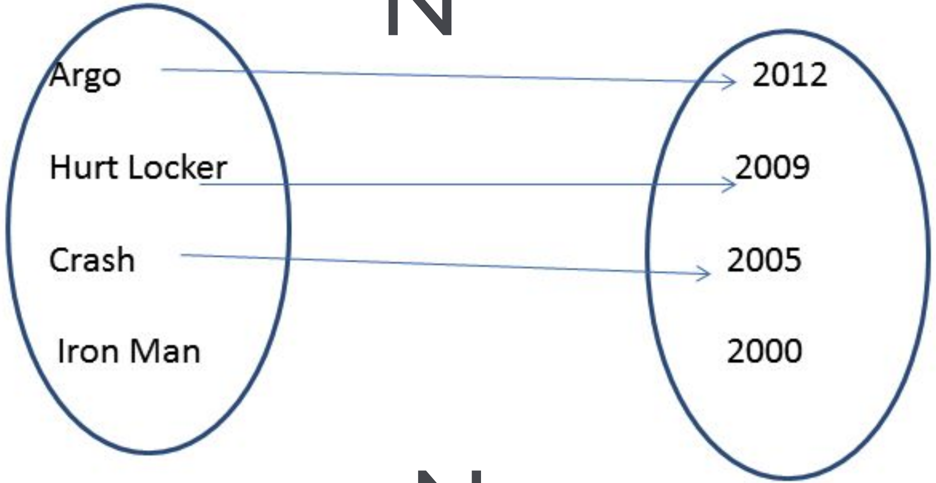
Review - Functions

Which of these are functions, and why?

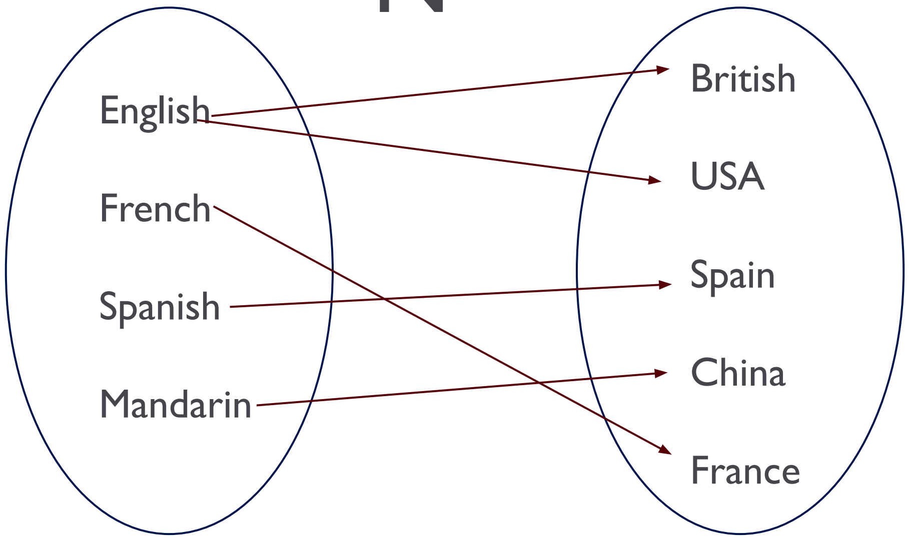


Y

N



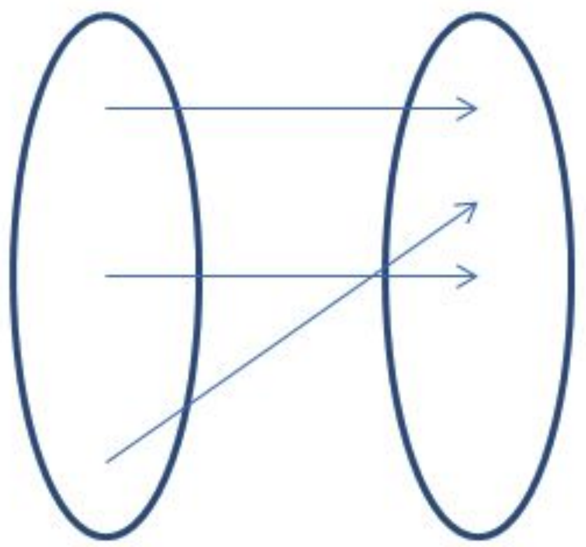
N



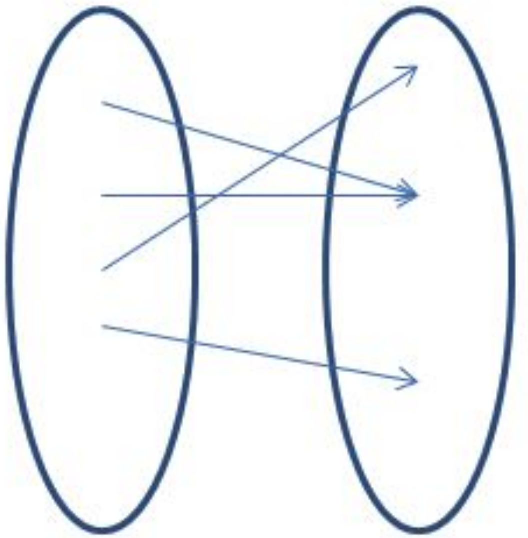
Review - One-to-One (injective)

A function f: A→B is called **injective** (or **one-to-one**) if **every element in the codomain B** is mapped by **at most one element in the domain A**.

if a1 maps to b1, then a2 cannot map to b1



One - one



Not one – one

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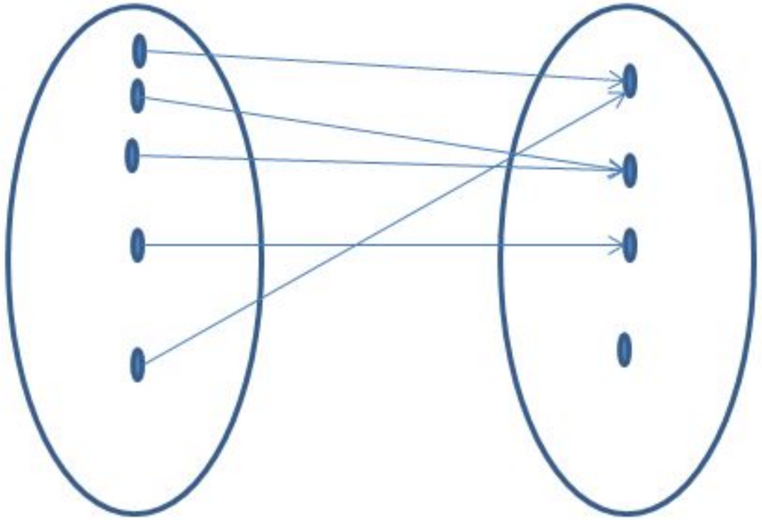
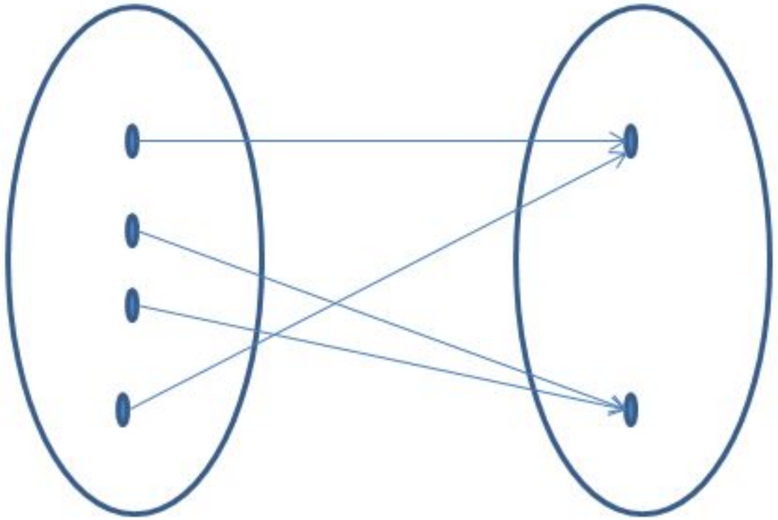
Review - Onto (surjective)

A **surjective** function is a function f such that, for every element y of the function's codomain, there exists at least one element x in the domain. In other words, every y must have an x. No element of the codomain can be left out.



Not onto
(also not one – one)

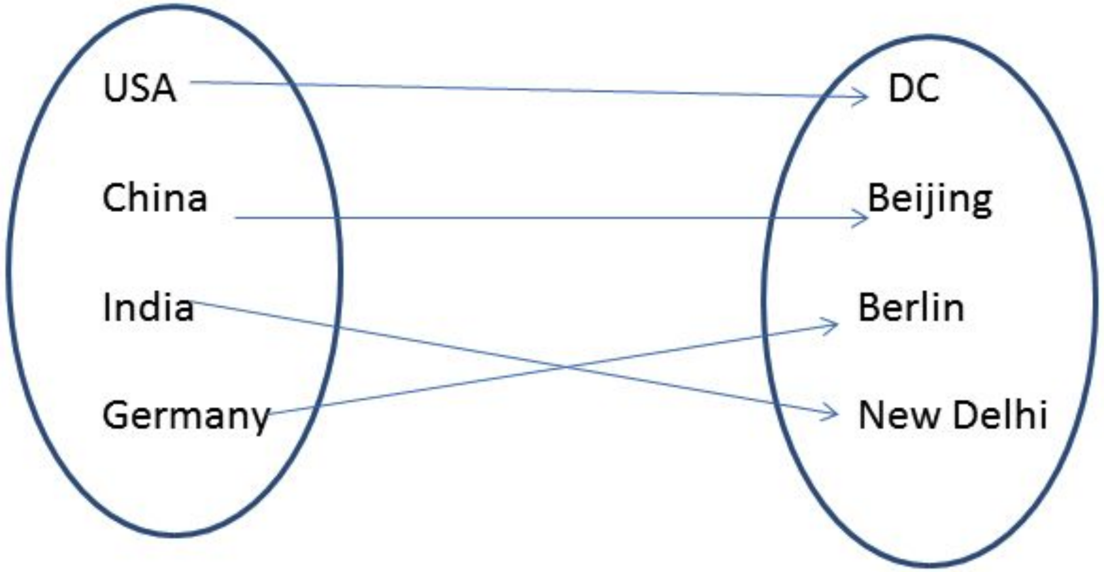


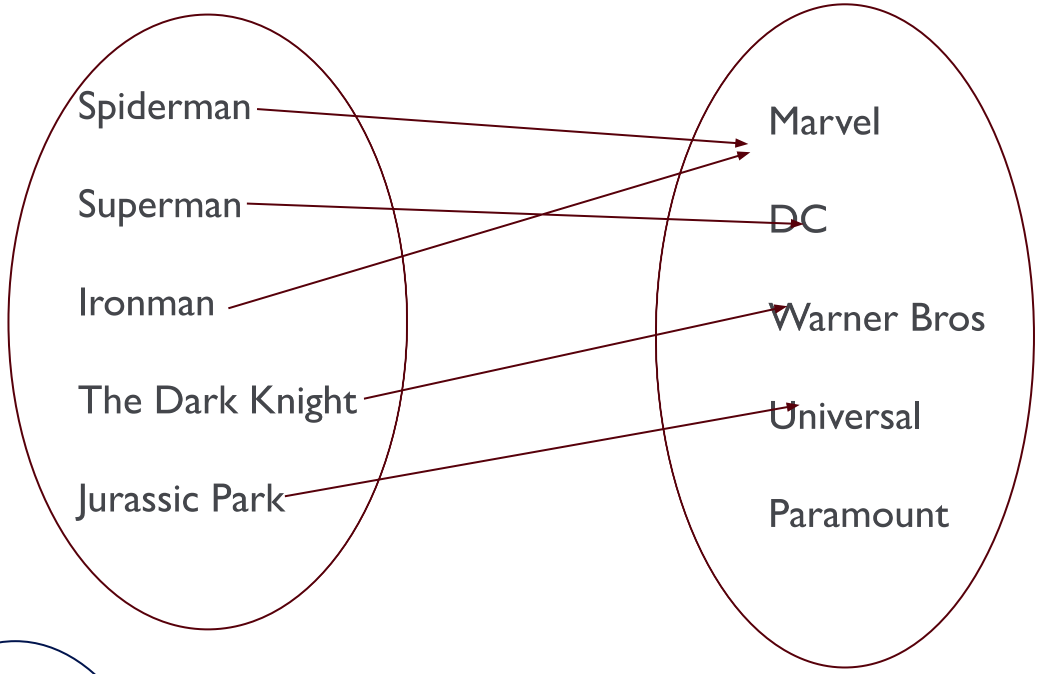
Onto
(this is not one-one though

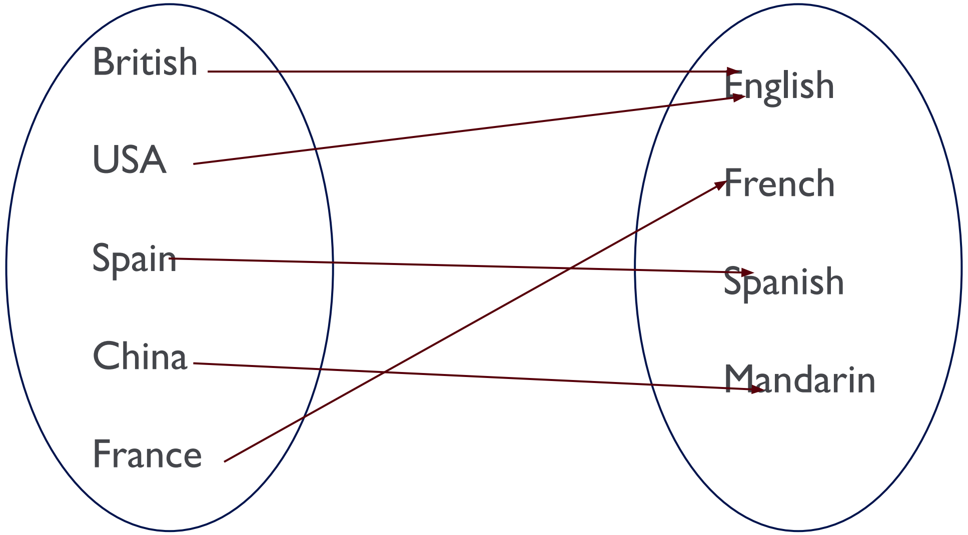
Review - Practice

Which of the following are functions? Injective? Surjective? Bijective?

Review - Practice





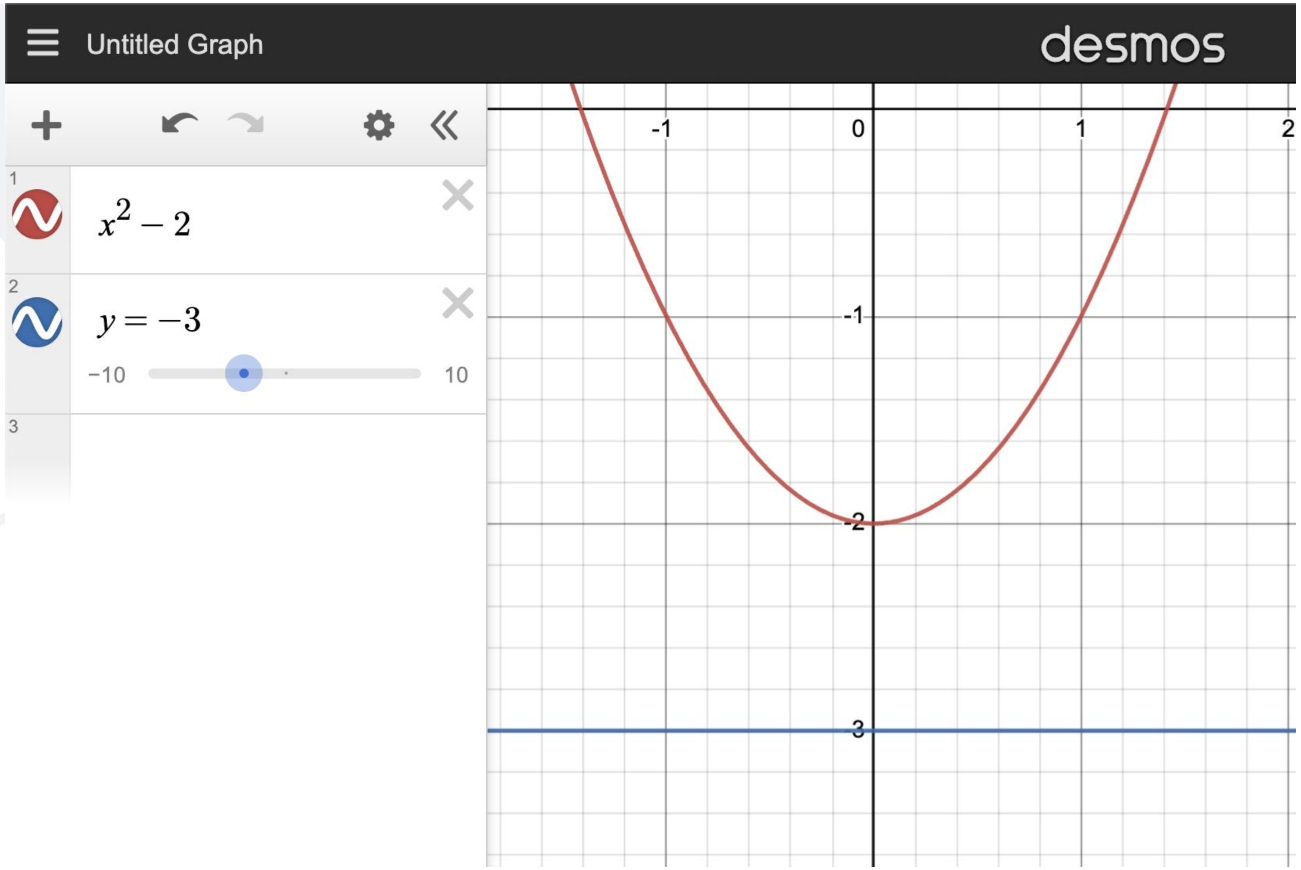


Question 1

Is $g(x) = x^2 - 2$ onto $g : \mathbb{R} \rightarrow \mathbb{R}$? Note that to show a function is not onto you have to produce one element in the co-domain that is not the output/target/image of the function for any argument/input to the function.

Answer 1

No, $g(x)$ is not onto. $-3 \in \mathbb{R}$, but -3 is not in the image of g.

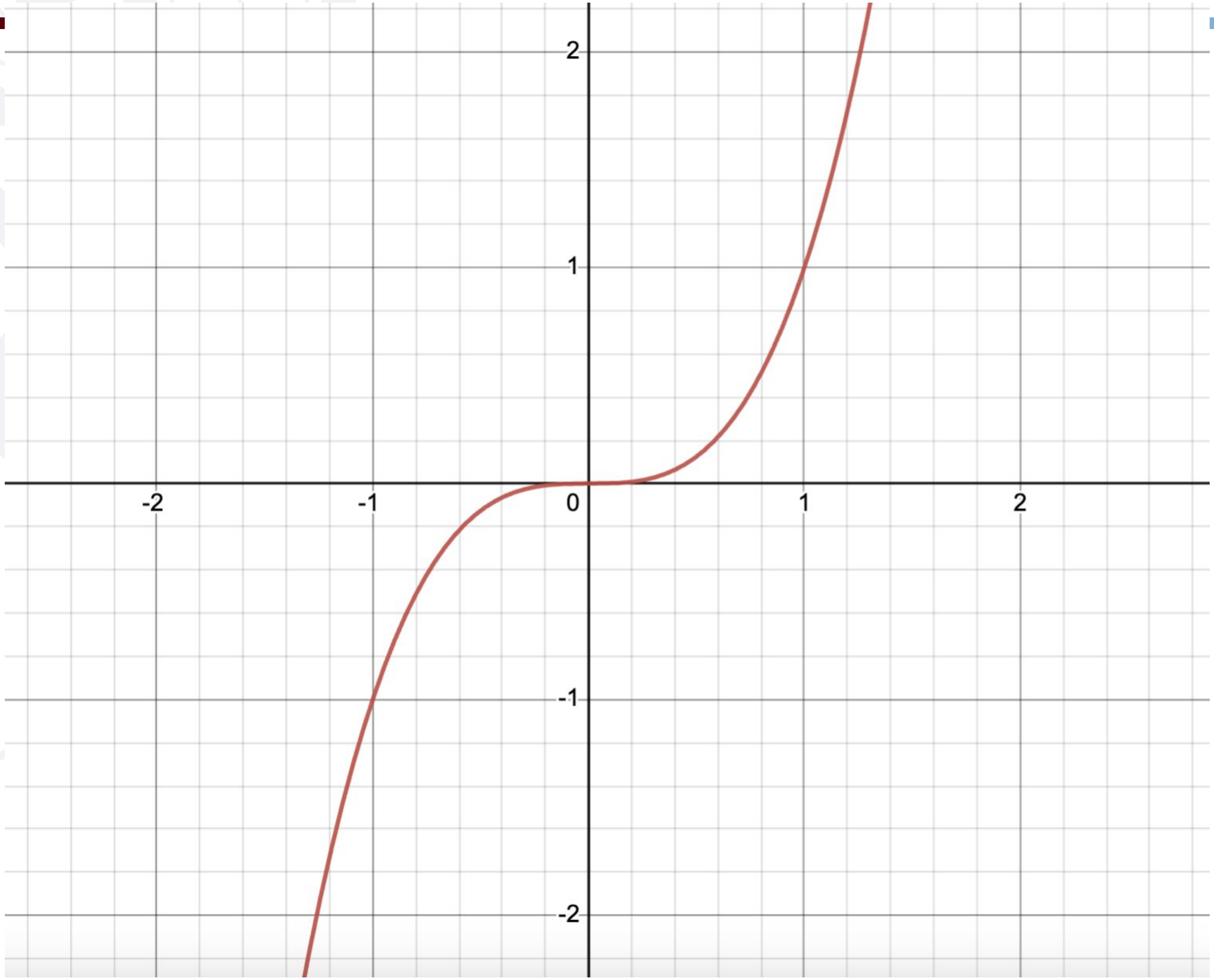


Question 2

Is $f(x) = x^3$ from the set of real numbers to real numbers one-one? Why or why not? Remember that a function is one to one if you can show that $f(x) = f(y)$ would imply that $x = y$.

Try to solve this algebraically,
but you may graph it for some intuition.

Answer 2



Answer 2 cont.

Suppose this function is not one-to-one. Then $\exists$ *some a,b* $\in \mathbb{R}$, $a \neq b$, *s.t. f(a)=f(b):*

$$\begin{aligned} a^3 &= b^3 \\ a^3 - b^3 &= 0 \\ (a-b)(a^2+ab+b^2) &= 0 \end{aligned}$$

If (a-b) is zero, then a=b and we have a contradiction. If the other term in parentheses is zero:

$$\begin{aligned} (a^2+ab+b^2) &= 0 \\ a^2+2ab+b^2 &= ab \\ (a+b)^2 &= 2ab \end{aligned}$$

Contradiction. f is one-to-one.

Question 3

Let A be the set $\{591, 592, 593\}$. Let B be the set of all subsets of students at Penn.

Define a mapping from A to B that maps a course in A to the students that are taking that course.

Is this mapping a function? Why or why not?

Is this function onto? Why or why not?

Is this function one to one? Why or why not?

State any assumptions you make while answering the question. Your assumptions should not be overly absurd.

Answer 3

I. Is a function

For one given course in set A, there should be a fixed number of students taking that class, and that collection of students must exist in set B because B is the set of all subsets of students at Penn. And because subsets should not have duplicates, for a given course in A, there should be only one element in B that is mapped to that course. Therefore, this mapping is a function.

II. No

Penn usually has thousands of students at least, so the size of set B should be much more than 3(the size of set A). In that case, it is not possible to have one course map to every element in B. Therefore it is not onto. (What would the answer be if there are only one student at Penn? It is an unrealistic assumption but fun to think about.)

III. Depends

In order to make the function one to one, we need to make sure every element in Set A is mapped to a different element in B.

We could have the case that, for CIT 591, 592, 593, they all have different collections of students enrolled, which would make this function one to one.

But it is also possible that two out of the three, or even all three courses, have the exact same collections of students enrolled, in that case, this function is not one to one.

Question 7

Assume we have a relation R defined on set A . The relation is reflexive, symmetric, and transitive (such relations are called **equivalence** relations). Now, for every element $b \in A$, we define $[b] = \{x | x \in A \text{ and } (x, b) \in R\}$.

First show that every $[b]$ is non-empty. Tell us why that set will contain at least one element.

What happens if $x \in [b]$ and that same $x \in [c]$? What can you say about $[b]$ and $[c]$ if they have an element in common?

Answer 7

Show that every $[b]$ is non-empty.

SOLN:

R is defined on the entire set A. Since R is reflexive, b∈A implies that (b,b)∈R. R is therefore non-empty for any non-empty set A.

What happens if $x \in [b]$ and that same $x \in [c]$? What can you say about $[b]$ and $[c]$ if they have an element in common?

SOLN:

R is defined s.t. x ∈[b] implies that (x,b)∈R.
Because R is symmetric, we also note that (b,x)∈R.
As with b, x ∈[c] implies that (x,c)∈R.
R is transitive, so (b,x)∈R and (x,c)∈R implies that (b,c) is also in R.
b,c∈A and (b,c),(c,b)∈R implies that c∈[b] and b∈[c]. some element p is in [b], it is related to b. By transitivity, is must also be related to c. Likewise, any q in [c] must be related to b. In other words, having some x in both [b] and [c] implies that ***any*** *element related to b must also be related to c.*
b and c are equivalent!

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Counting

Recall- The product rule

If Set A has m elements and Set B has n elements, the Cartesian product $A \times B$ has $m \times n$ pairs.

Example:

- A = {1, 2}, B = {x, y}
- A × B = {(1, x), (1, y), (2, x), (2, y)}
- Total pairs = 2 × 2 = 4

Counting (Product Rule)

Similarly- The product rule is connected to the Cartesian product of sets

The Product Rule (Multiplication Rule):

- If one event can occur in m ways and another independent event can occur in n ways, the total number of ways both events can occur together is $m \times n$.
- **Example**:
  - You have 3 shirts and 2 pants.
  - Total outfits = 3 (shirts) × 2 (pants) = 6 outfits.
  -

Counting(Product Rule No Repeats)

If we have no repeats, we must remove the item from the one set of the cartesian product.

I have 5 books. How many ways can I arrange 3 of them on a shelf?

After choosing 1 of the 5 books, then we have to choose 1 of the remaining 4, then 1 of the remaining 3.

$\underline{5} \times 4 \times \underline{3} = 60$

Counting (Sum Rule)

The Sum Rule (Addition Rule):

- If one event can occur in *m* ways and a different, mutually exclusive event can occur in *n* ways, the total number of ways either event can occur is $m + n$.
- $|A \cup B| = |A| + |B|$

Two events are **mutually exclusive** if :

- They cannot occur at the same time (if one event occurs, the other cannot.)

- **Example**:
  - You have 4 types of sandwiches or 3 types of salads.
  - Total meal options = 4 (sandwiches) + 3 (salads) = 7 options.

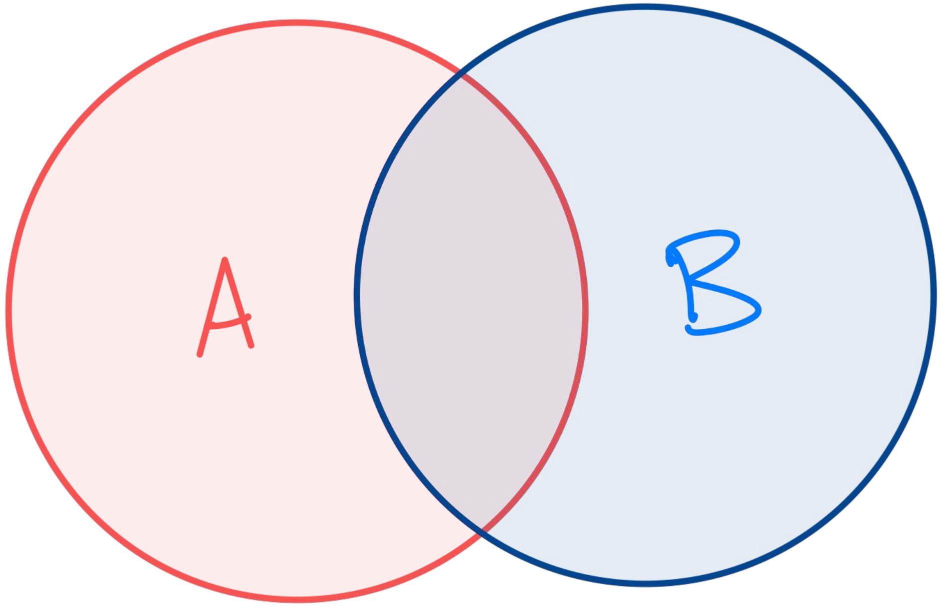
Counting(Inclusion Exclusion)

If events are **not mutually exclusive** we use the Inclusion Exclusion Principle:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

When events are ı ounts

$|A| + |B|$ $|A \cap B|$



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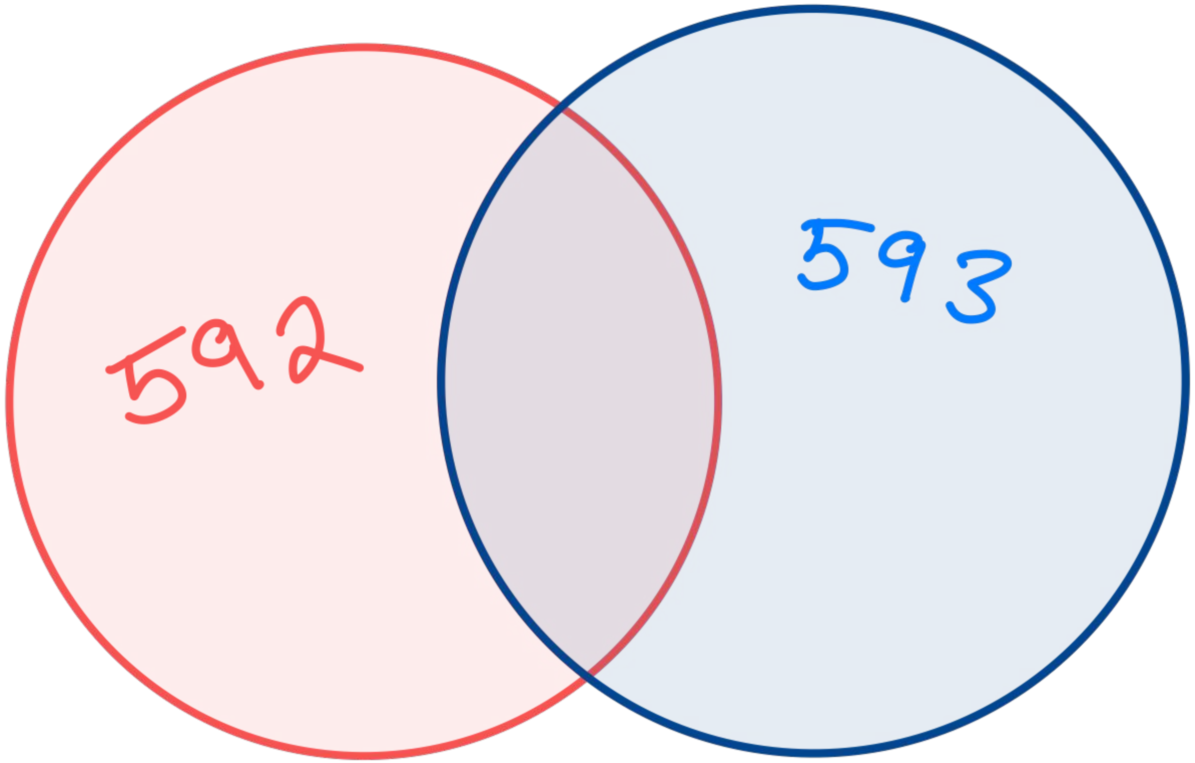
Inclusion Exclusion Example

In a group of 60 students:

- 25 students take **CIT 5920**.
- 30 students take **CIT 5930.**
- 15 students take **both CIT 5920 and CIT 5930**.

Question: How many students study either or both?

$$|593 \cup 592| = 25 + 30 - 15 = 40$$



Question 4 (with Repetitions)

How many three digit numbers between 100 and 500 (both included) can be formed from the digits 1, 2, 3, 4, 5, 6, and 7 **with repetitions** allowed?

Answer 4 (with Repetitions)

For the first digit, we can either have 1, 2 , 3, or 4.

For the second digit, we can choose 1, 2, 3, 4, 5, 6, 7

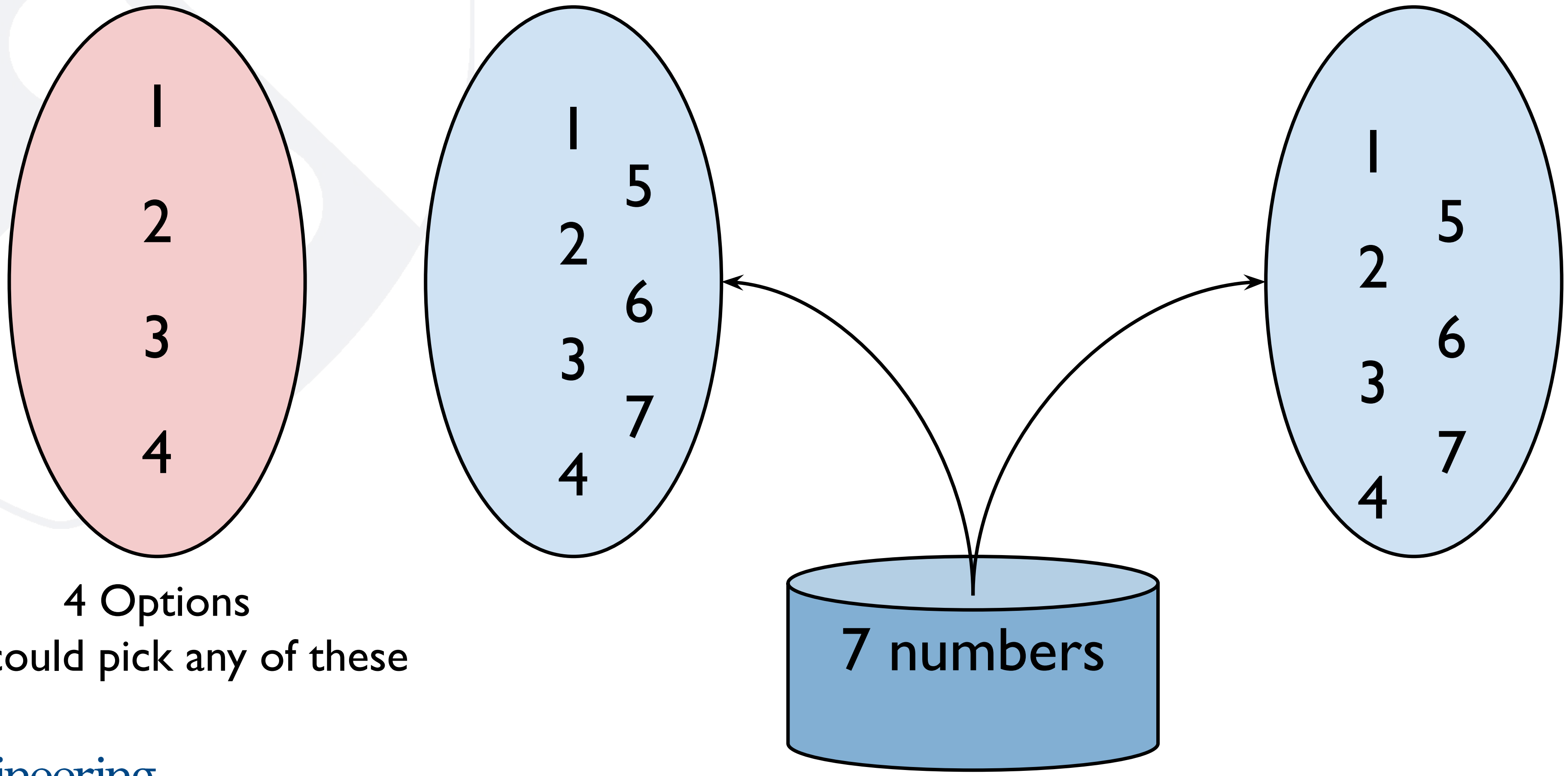
For the third digit, we can choose 1, 2, 3, 4, 5, 6, 7

For this we have the cartesian product |1st| x |2nd| x |3rd|

Or $\underline{4} \times \underline{7} \times \underline{7} = 196$

Answer 4

These digits will give us numbers between 100 and 500



4 Options
We could pick any of these

Question 4 (without Repetitions)

How many three digit numbers between 100 and 500 (both included) can be formed from the digits 1, 2, 3, 4, 5, 6, and 7 **with NO repetitions** allowed?

Answer 4 (without Repetitions)

Solution:

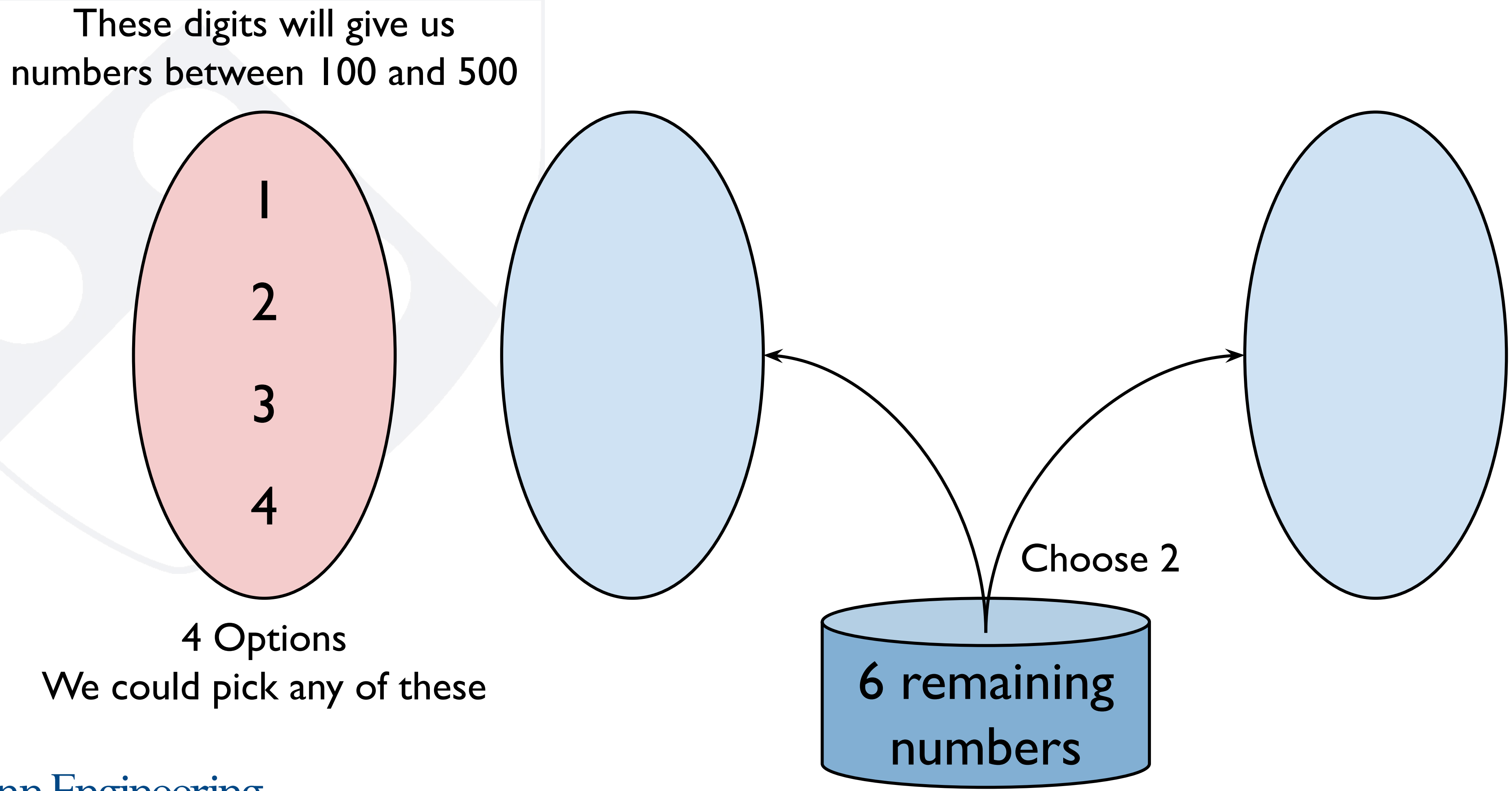
In order to make sure the three digit number is between 100 and 500, we can only choose 1, 2, 3, or 4 for the first digit. So we have 4 choices for this one.
Then we choose the next two digits, we do not need to have any anymore constraints, therefore, we can free to choose two out of the rest 6 digits. And this step is P(6, 2)
Overall, we have: $4 \times P(6, 2)$

No longer a cartesian product A x A x A

Now we have X * (N-1) * (N - 2)

4 * 6 * 5 =120

Answer 4


These digits will give us
numbers between 100 and 500
1
2
3
4
4 Options
We could pick any of these
Choose 2
6 remaining
numbers

Question 5

Exercise 2

A palindrome is a word that reads the same forward and backward. How many seven letter palindromes can be made in the English language?

* Doesn't have to be an actual word
* Include capital letters

Answer 5

Solution:

$$52^4$$

Consider an array ”a” of 7 that represent this 7 letter palindromes.
We know:

$$a[0] = a[6]$$

$$a[1] = a[5]$$

$$a[2] = a[4]$$

$$a[3] = a[3]$$

Bounded by the rules above, once we have selected the first 4 letters, the whole string will be generated.
Consider we have lowercase and uppercase letters, but lowercase ’a’ and uppercase ’A’ are consider as the same letter.
We have 52 choices for each position. Since there are 4 positions and they don’t have to be unique, the answer should be: 52^4

Question 6

Exercise 4

Arvind's uncle has two dogs named Ozzie and Lena. Arvind has a dog named Lucy. When they get together they put down 3 bowls of food. The bowls are numbered 1, 2, and 3.

Once a dog starts eating the food in a bowl, it will finish all the food, and will not allow any of the other 2 dogs to get a piece.

In how many ways can these dogs consume the food? A dog is allowed to go hungry. A dog is also allowed to eat all 3 bowls (if the other 2 dogs are being lazy, this will happen).

To give you some examples of what the "ways" look like

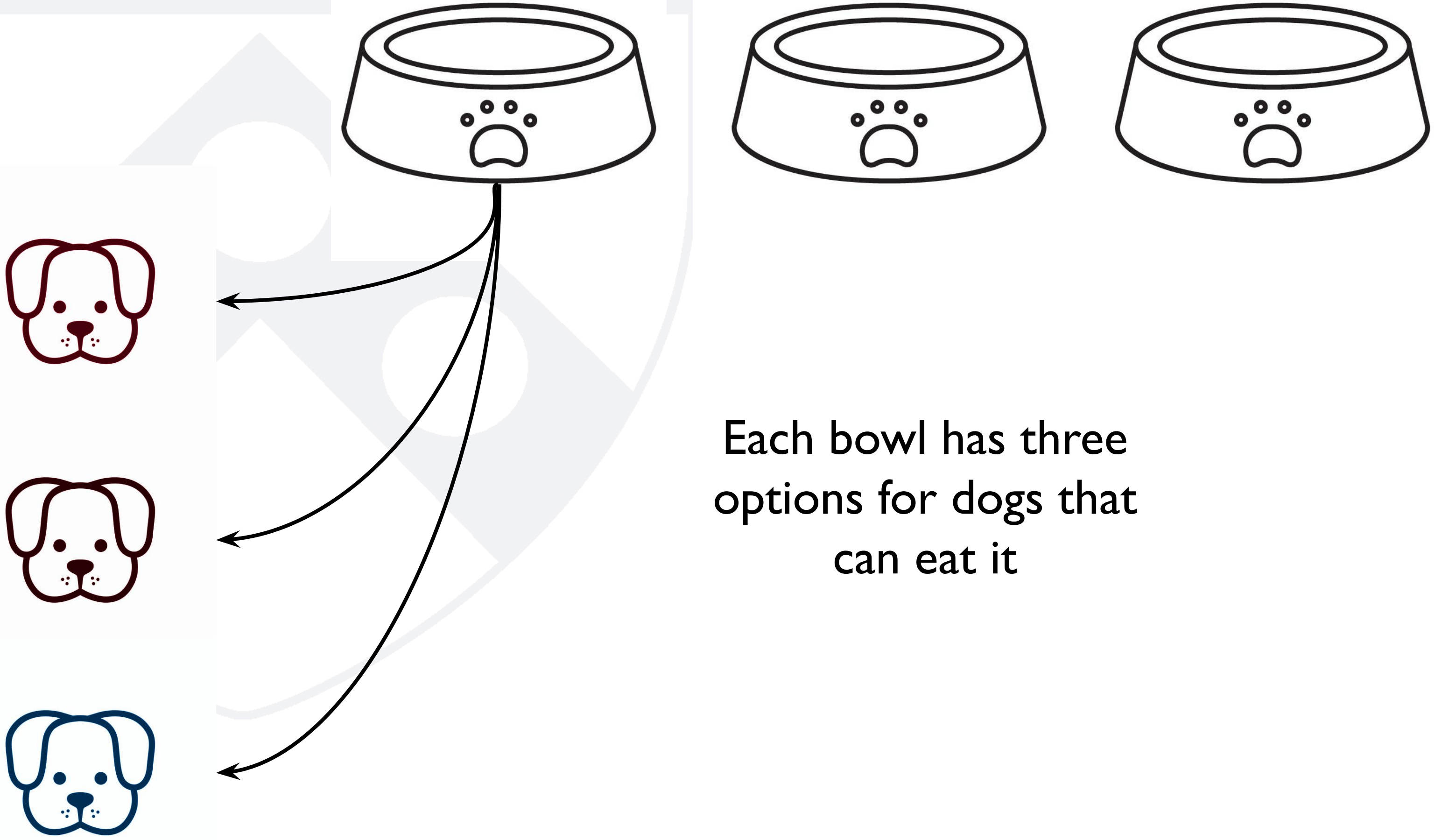
Lucy ate bowl 1, bowl 2, and bowl 3

is different from

Lena ate bowl 1, Ozzie ate bowl 2, Lucy ate bowl 3.

Note: Assume that each bowl must be eaten by the end, i.e. each bowl must be eaten by exactly one dog

Answer 6



Each bowl has three options for dogs that can eat it

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Answer 6 (cont.)

Solution:

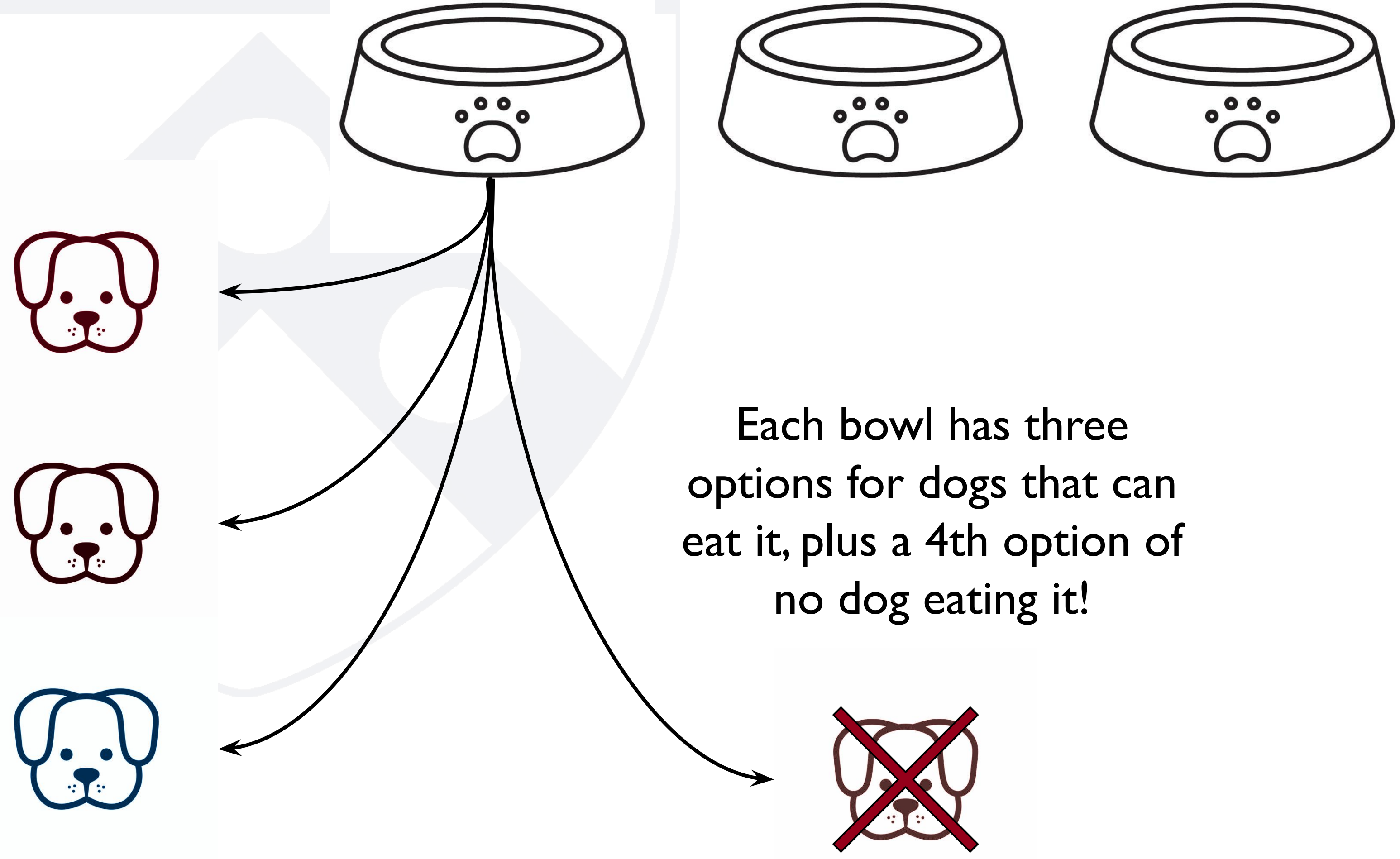
$$3^3$$

Each bowl has potentially 3 dogs that could eat it.

Follow-up Question

What if there was no requirement that every bowl must be eaten?

Follow-up Answer



Each bowl has three options for dogs that can eat it, plus a 4th option of no dog eating it!

Solution: 4^3

Question 7

Exercise 5

Please answer the following questions about 5 digit numbers.

A. How many 5 digit numbers are there?

B. How many 5 digit numbers have no two consecutive digits equal?

C. How many have at least 1 pair of consecutive digits equal?

Note: An "n"-digit number must start with something non-zero, otherwise it wouldn't be "n" digits

Answer 7

Solution: 9×10^4
There are 10 digits from $0 - 9$. For a, the first digit of a number cannot be 0, so there are 9 options for the first digit and 10 for the remaining 4 digits, resulting in 9×10^4.

Solution: 9^5
There are still only 9 options for the first digit; there are only 9 options for the remaining 4 digits since they cannot be the same as the previous.

Solution: $(9 \times 10^4) - 9^5$
Part C is just the difference between the results of A and B because to have at least 1 pair of consecutive digits equal is also to have no numbers where no two consecutive digits are equal.

Bonus Question

Exercise 3

How many strings of length 4 can be formed using the English alphabet in the following cases? Both upper case and lower case letters allowed. No explanation needed.

A. Repeats are allowed.

B. No letter can be repeated. We will consider something like "Babs" an example of repeated letters.

We can use both uppercase and lowercase letters, but once we use say an uppercase B, both the uppercase and lowercase versions of b are now unavailable to us

Bonus Question Answer A

Solution: With Repeats: 52^4

String of length 4: □ □ □ □

$$52 \times 52 \times 52 \times 52 = 52^4$$

Since both uppercase and lowercase letters are allowed, the total number of available letters is 52 (26 uppercase + 26 lowercase). Since repeats are allowed, for each position, you have 52 possible choices. The letters in one position does not affect the choices in other positions.

To find the total number of possible strings, you multiply the number of choices for each position together. This is an example of the **product rule**, where each position is independent of the others.

Bonus Question Answer B

Solution: No Repeats: $52 \times 50 \times 48 \times 46$
For each letter following the first, we remove the uppercase along with the lowercase of that letter.

String of length 4: □ □ □ □

$52 \times 50 \times 48 \times 46$

The first position can hold any of the 52 letters (uppercase or lowercase).

After you have chosen a letter for position 1, you cannot use that same letter (uppercase or lowercase) for any of the remaining positions (no repeats).

This means, for position 2, you have $52 - 2 = 50$ options, for position 3, you have $50 - 2 = 48$ options, and for position 4, you have $48 - 2 = 46$ options.

Question 6

Exercise 6

How many diagonals does an n sided polygon (assume it is a convex polygon) have?

Answer 6

Solution:

$$\binom{n}{2} - n = \frac{n(n-3)}{2}$$

An n sided polygon has n vertices. A diagonal is formed by 2 vertices which can be formed in $\binom{n}{2}$ ways. This number also includes the sides of the polygon, so we subtract n from the total.

Basically, you can think of a line in a polygon as having to choose two points out of n points to connect. But, since we only want diagonals, we have to ignore the n lines that already exist to form the boundary of the polygon

Question 7

Exercise 7

Suppose that a pet shelter contains 10 dogs and 15 cats. Harry's brother loves dogs and his sister-in-law loves cats. They would like to adopt 5 animals. In how many ways can they get 5 pets with the constraint that they have to have more cats than dogs?

Answer 7

Solution:

$$\binom{10}{0}\binom{15}{5}+\binom{10}{1}\binom{15}{4}+\binom{10}{2}\binom{15}{3}$$

This problem can be divided into the following three cases to adopt *at least* 3 cats:

1) 0 dogs, 5 cats: $\binom{10}{0}\binom{15}{5}$

2) 1 dog, 4 cats: $\binom{10}{1}\binom{15}{4}$

3) 2 dogs, 3 cats: $\binom{10}{2}\binom{15}{3}$

Question 8

Exercise 8

There are ten points on a plane. 4 of these points are in a straight line and with the exception of these four points, no other three points are in the same straight line. How many distinct straight lines can be drawn such that they pass through at least 2 of these points?

Answer 8

Solution:

$$\binom{10}{2} - \binom{4}{2} + 1$$

Among any 10 points, there are up to $\binom{10}{2}$ segments between them (# of ways to draw a line between pairs of points). However, in this case we already know that four of them occupy a single line, thus we subtract out all of the combinations that those four points could have ($\binom{4}{2} = 6$) and just add back their one shared line: $\binom{10}{2} - \binom{4}{2} + 1$

Question 8

Exercise 9

How many bit strings of length ten either start with 000 or end with 1111?

Answer 8

Solution:

$$2^7 + 2^6 - 2^3$$

The key concept here is the idea that $A \cup B = A + B - A \cap B$ applies to counting problems.
If we just wanted all ten-bit strings of 0s and 1s, we would have 2^{10} possibilities.
To count the strings that start with 000, we just assume those digits are fixed and now have only 2^7 possibilities.
Similarly, there are just 2^6 ending in 1111.
Don't forget that these two collections overlap: we are double-counting strings that *both* begin with 000 and end with 1111.
Let's subtract these overlapping items out of our final count:
$2^7 + 2^6 - 2^3$ 10-digit bit strings begin with 000, end with 1111 or both.

Homework Question Poll

Functions

Counting

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See you next week! <33333

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