Lecture 11: More Stars and Bars

STANDARD VERSION

How many ways to hand out exactly 15 marbles to 3 children?

15 marbles → units to distribute (non-distinct)

3 children ↳ categories (distinct)

the number of marbles received by the first child

$$\boxed{x_1 + x_2 + x_3 = 15}$$

marbles of 2nd child

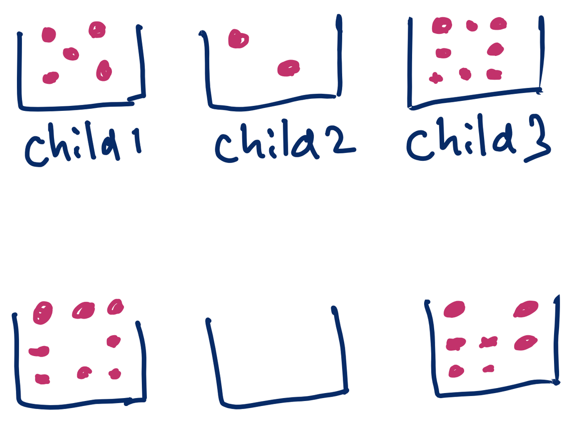
Count the number of solutions

$$\binom{15+3-1}{3-1}$$

$$\binom{A+B}{A} = \binom{A+B}{B}$$

$$\frac{(A+B)!}{A!\,(A+B-A)!} \qquad \frac{(A+B)!}{B!\,(A+B-B)!} = \frac{(A+B)!}{A!\;B!}$$

A sample solution to the allocation problem



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we are counting how many valid words can be formed ⚠

EQUATIONS WITH $\leq T$ (upperbound)

How many ways to hand out at most 15 marbles to 3 children?

$$x_1 + x_2 + x_3 \leq 15$$

Introduce an extra variable (called a "slack variable")

child 1, child 2, child 3, Leftover x_4

$$x_1 + x_2 + x_3 + x_4 = 15$$

Then we solve like before

$$\binom{15+4-1}{4-1}$$

4 because, with the slack variable we now have 4 categories (instead of 3)

"contains all marbles not given to child 1, 2, 3"

Quiz How many ways to hand out at most 14 marbles to 3 children? thx Jacob!

$$\binom{14+4-1}{4-1} = \binom{17}{3} = 680$$

Best — Good — Good (but trying too hard)

EQUATIONS WITH $\geq$ (LOWER BOUND) thx Jerry!

How many ways to distribute at least 15 marbles to 3 children?

$$x_1 + x_2 + x_3 \geq 15$$

↳ the number of solutions is infinite because you have an infinite supply of marbles

To solve this problem, we need an UPPERBOUND on the x_i variables such as

x_1, x_2, x_3

$$\begin{cases} x_1 \leq 7 \\ x_2 \leq 7 \\ x_3 \leq 7 \end{cases}$$

⚠ (when each child can receive at most 7 marbles)

with these additional constraints, the max upper bound would be if all x_i took their LARGEST VALUE, i.e.

$$7 + 7 + 7 = 21$$

So we would then ADD UP the number of ways of solving each equation

$x_1 + x_2 + x_3 = 15$
OR $x_1 + x_2 + x_3 = 16$
OR $x_1 + x_2 + x_3 = 17$
OR $x_1 + x_2 + x_3 = 18$
OR $x_1 + x_2 + x_3 = 19$
OR $x_1 + x_2 + x_3 = 20$
OR $\boxed{x_1 + x_2 + x_3 = 21}$

Great

$$\text{Total ways} = \binom{15+3-1}{3-1} + \binom{16+3-1}{3-1} + \binom{17+3-1}{3-1} + \binom{18+3-1}{3-1} + \dots + \binom{21+3-1}{3-1}.$$

$$= \sum_{i=15}^{21} \binom{i+3-1}{3-1}$$

← Also great

VARIABLES WITH ADDITIONAL CONSTRAINTS

CONSTRAINTS OF THE FORM $x_i \geq A$

How many ways of distributing 15 marbles to 3 children if each child gets at least 3 marbles?

$$\begin{cases} x_1 + x_2 + x_3 = 15 \\ x_1 \geq 3 \\ x_2 \geq 3 \\ x_3 \geq 3 \end{cases}$$

subtract lowerbounds from units to distribute

$3+3+3 = 9$

$$\begin{cases} (x_1 - 3) + (x_2 - 3) + (x_3 - 3) = (15 - 9) \\ x_1' + x_2' + x_3' = 6 \\ x_1' \geq 0 \quad x_2' \geq 0 \quad x_3' \geq 0 \end{cases}$$

Thus we can solve our original problem with

$$x_1' + x_2' + x_3' = 6 \implies \binom{6+3-1}{3-1} = \binom{8}{2}$$