Lecture 10

Quiz

1. $5 \times 4 \times 3 \times 2 \times 1 = 5! = \boxed{120}$

2. difficulty: two indistinguishable P's

   APPLE $\equiv$ APPLE$_{2,1}$ (the P's are indistinguishable — we need to count APPLE$_2$ and APPLE$_{2,1}$ as the same)

   we can swap P_1 and P_2 → $\frac{5!}{2} = \frac{120}{2} = \boxed{60}$

3. without repetition

   $3 \times 2 \times 1 = 3! = \boxed{6}$

   10 possibilities

   with repetition

   $3 \times 3 \times 3 = 3^3 = \boxed{27}$

6. $6 \times 5 \times 4$

ARRANGEMENTS with NON-DISTINCT OBJECTS

Def When arranging objects where some are IDENTICAL, the standard permutation formula $n!$ (which counts the arrangements of n distinct objects) OVERcounts the number of unique arrangements

e.g. APP_2 →

$A P_1 P_2$ — here $3! = 6$ is an overcount

$P_1 A P_2$

$P_1 P_2 A$

$A P_2 P_1$

$P_2 A P_1$

$P_2 P_1 A$

only 3 ways of shuffling these letters

all of these are duplicate words (if we consider $P_1 = P_2$)

We need to adjust for the indistinguishable items.

FORMULA for PERMUTATIONS with REPETITION

If there are n total objects, where

- k_1 objects are of type 1
- k_2 objects are of type 2
- k_3 objects are of type 3
- ...
- k_m objects are of type m

with $k_1 + k_2 + \dots + k_m = n$

number of distinct arrangements

$$\frac{n!}{k_1! \, k_2! \, \dots \, k_m!}$$

- identical objects cannot be distinguished; swapping does NOT create a new arrangement
- adjustment: we divide by the factorial of the counts of each type to compensate for overcounting

EXAMPLE 1: "BALLOON" how many unique ways to arrange the letters of this word?

Letters and Counts
- B : 1
- A : 1
- L : 2
- O : 2
- N : 1

$n = 1 + 1 + 2 + 2 + 1 = 7$

$$\frac{n!}{k_B! \times k_A! \times k_L! \times k_O! \times k_N!} = \frac{7!}{1! \times 1! \times 2! \times 2! \times 1!} = \frac{7!}{2! \times 2!} = \boxed{\frac{7!}{4}}$$

EXAMPLE 2: Arranging beads of different colors.
We have 10 beads: 4 red, 3 blue, 3 green.
How many necklaces can you make by stringing all the beads in a line.

$$\boxed{\frac{10!}{4! \times 3! \times 3!}}$$

EXAMPLE 3: "MISSISSIPPI" ? $n = 11$ letters

$$\frac{11!}{k_M! \times k_I! \times k_S! \times k_P!} = \frac{11!}{1! \times 4! \times 4! \times 2!}$$

COUNTING INTEGER SOLUTIONS ("STARS and BARS" method)

Counting the number of non-negative integer solutions to equations of the form:

$$x_1 + x_2 + x_3 + \dots + x_n = k$$

is a common problem in combinatorics. This type of problem can be solved using the Stars and Bars method, which relates to combinations with repetition.

- Non-negative integers values $x_i \geq 0$ ("x sub i")
- Stars and Bars: A visual method to represent the distribution of identical items into distinct categories

STARS AND BARS method

Def
- STARS $*$ represent the units to be distributed (eg candy)
- BARS $|$ represent the dividers between categories (e.g. people)

Total symbols
- number of stars = k (total units to distribute)
- number of bars = $n - 1$ (dividing the units among n variables)

Formula

$$\text{number of (integer) solutions} = \binom{k + n - 1}{n - 1} = \left[\frac{(k+n-1)!}{(k+n-1-(n-1))! \, (n-1)!} \right]$$

the Jeeaj!

EXAMPLE 1 How many ways can you distribute 10 identical candies to 4 (different) children so that each child can receive any number of candies. (including zero)

- Variables $x_1 \quad x_2 \quad x_3 \quad x_4$ ↑ number of candies received by child 1 (we number the children — we could give them names)
- Equation: $x_1 + x_2 + x_3 + x_4 = 10$
- Applying Stars and Bars
  - STARS : 10 candies
  - BARS : $4 - 1 = 3$ dividers

  ] 13 symbols in total

$$\text{number of solutions} = \binom{10 + 4 - 1}{4 - 1} = \binom{13}{3} = 286$$

→ the different "WORDS" we can form with stars and bars

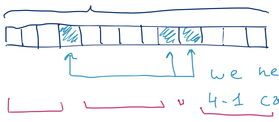
What are we counting — ONE POSSIBLE COMBINATION WITH REPETITION

`***|**|*****|`

child 1 receives 3 candies; child 2 receive 2 candies; child 3 receives 5 candies; child 4 receives 0 candies

`||||**********`

children 1, 2, 3 receive 0 candy; child 4 receives 10 candies

ENCODING

Let's take a closer look $\binom{10 + 4 - 1}{4 - 1}$