Lecture 9A: More counting examples

STANDARD STARS AND BARS (exactly)

How many ways to hand out 15 marbles to 3 children?

units to distribute

↳ categories

$$\boxed{x_1 + x_2 + x_3 = 15}$$

Number of ways

$$= \binom{15+3-1}{3-1} = \binom{17}{2}$$

EQUATION WITH ≤

How many ways to handout at most 15 marbles to 3 children?

$$\boxed{x_1 + x_2 + x_3 \leq 15} \quad (*)$$

Introduce an extra variable (called a "slack variable")

$$x_{n+1} = 15 - [x_1 + x_2 + x_3]$$

which allows us to rewrite (*) as:

$$x_1 + x_2 + x_3 + x_4 = 15$$

will contain all marbles that are not handed out [to children 1, 2 or 3]

We can now apply S&B: $\binom{15+3+1-1}{3+1-1} = \binom{18}{3}$

Quick What about: How many ways to handout at most 14 marbles to 3 children?

+1 comes from add. slack variable

$$\binom{18}{3} - \binom{17}{2} \quad \text{or} \quad \binom{14+3+\boxed{+1}-1}{3\boxed{+1}-1} = \binom{17}{3}$$

$$\boxed{\binom{18}{3} - \binom{17}{2} = \binom{17}{3}}$$

EQUATION WITH ≥ (upper bound)

How many ways to distribute at least 15 marbles to 3 children?

$$\boxed{x_1 + x_2 + x_3 \geq 15}$$

→ the number of solutions is infinite because you have an infinite number of marbles

To be able to solve this question we would need upper-bound constraints on the x_i, such as

$$x_1, x_2, x_3 \leq 7$$

with the additional constraint, the max upperbound would be if all x_i took their largest value, i.e.

$$7 + 7 + 7 = 21$$

So we would ADD UP the number of ways of solving each of these equations:

$$x_1 + x_2 + x_3 = 15$$
$$x_1 + x_2 + x_3 = 16$$
$$x_1 + x_2 + x_3 = 17$$
$$x_1 + x_2 + x_3 = 18$$
$$x_1 + x_2 + x_3 = 19$$
$$x_1 + x_2 + x_3 = 20$$
$$x_1 + x_2 + x_3 = 21$$

Total ways =

$$\binom{15+3-1}{3-1} + \binom{16+3-1}{3-1} + \binom{17+3-1}{3-1} + \binom{18+3-1}{3-1} + \dots + \binom{21+3-1}{3-1}$$

VARIABLES WITH ADDITIONAL CONSTRAINTS

CONSTRAINT OF THE FORM $x_i \geq A$

How many ways of distributing 15 marbles to 3 children if each child gets at least 3 marbles.

$$\begin{cases} x_1 + x_2 + x_3 = 15 \\ x_1 \geq 3 \\ x_2 \geq 3 \\ x_3 \geq 3 \end{cases}$$

→ subtract lower bounds from the units to distribute

$$\begin{cases} \underbrace{(x_1 - 3)}_{x_1'} + \underbrace{(x_2 - 3)}_{x_2'} + \underbrace{(x_3 - 3)}_{x_3'} = (15 - 9) \\ x_1' + x_2' + x_3' = 6 \\ x_1' \geq 0 \\ x_2' \geq 0 \\ x_3' \geq 0 \end{cases}$$

Thus the number of ways is the solutions to:

$$x_1' + x_2' + x_3' = 6 \rightarrow \binom{6+3-1}{3-1} = \binom{8}{2}$$