Lecture 8 : Intro to counting

Question sample

- "How many possible passwords can you create with specific requirements?"
- "How many different ways to shuffle (mix) a playlist of your songs?"
- "How many different pizza combinations can you order?"

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1. The product rule

Sometimes when counting, we are making a series (sequence) of choices:

- we select option 1 then
- we _____ 2 then
- we _____ 3

In this situation, the product rule is useful. The product rule comes from the rule on Cartesian Products $|A \times B| = |A| \times |B|$

LITTLE REFRESHER ON CARDINALITY

$A = \{1, 2\}$ $B = \{x, y\}$

$A \times B = \{(1,x), (1,y), (2,x), (2,y)\}$

$|A| = 2$ $|B| = 2$

$|A \times B| = 4 = |A| \times |B|$

Example 1 Creating a sandwich

- 3 types of bread: {white, wheat, Rye}
- 4 types of filling: {tofu, chicken, falafel, ham}

How many different sandwiches can we make? $3 \times 4 = 1$

Example 2 Number of outfits

- 4 different shirts
- 3 different pants
- 2 different shoes

} total number of outfits $4 \times 3 \times 2 = 24$ different possible outfits

2. The Sum Rule

if instead of making choices one after the other, you are counting something where you have to choose between multiple options (choice 1 OR choice 2) then the sum rule is applied.

Example 1 The students in this classroom are divided into these three groups

Male + Female + Non-Binary = Students

because the sets are disjoints.

Example 2 The total number of North American citizens is the union of:

U = US citizens
C = CAnadian citizens
M = Mexican citizens

We can assume all North American citizens are counted by $|U| + |C| + |M|$ if we assume the sets are disjointed

Example 3 When you go to a restaurant and it serves:

- chicken
- ham
- tofu
- seitan

} 4 choices

You are eating one or the other, so 4 possible choices

EXAMPLES USING BOTH RULES

PASSWORD CREATION EXAMPLE

How many passwords can be made:

- just UPPER and lower case letters
- between 4 and 8 characters

password conditions

P = set of such passwords

$\mathcal{A} = \{\underbrace{A, B, C, \dots, Z}_{\text{UPPER CASE LETTERS (26)}}, \underbrace{a, b, c \dots, z}_{\text{lower case letters (26)}}\}$ $|\mathcal{A}| = 26 + 26 = 52$

P_k = set of passwords of length k

then, because the sets P_k are disjoint (because elements of P_i and P_j are of different lengths when $i \neq j$)

$P = P_4 \cup P_5 \cup P_6 \cup P_7 \cup P_8$

By the sum rule because the P_i are disjoint

$|P| = |P_4| + |P_5| + |P_6| + |P_7| + |P_8|$

How do we calculate $|P_k|$?

$P_4 = \mathcal{A} \times \mathcal{A} \times \mathcal{A} \times \mathcal{A}$ ← the elements in P_4 are sequences of 4 choices of letters

1st letter, 2nd letter, 3rd letter, 4th letter

By the product rule, we have

$|P_4| = |\mathcal{A} \times \mathcal{A} \times \mathcal{A} \times \mathcal{A}| = |\mathcal{A}| \times |\mathcal{A}| \times |\mathcal{A}| \times |\mathcal{A}|$

$= |\mathcal{A}|^4 = 52^4$

Putting it all together

$|P| = |P_4| + |P_5| + |P_6| + |P_7| + |P_8|$

$= \boxed{52^4 + 52^5 + 52^6 + 52^7 + 52^8}$.

- just UPPER LETTER + digits
- between 10 and 20 characters

The changes here are

$\mathcal{A}' = \{\underbrace{A, B, C \dots Z}_{\text{UPPER CASE (26)}}, \underbrace{0, 1, 2, 3, 4, 5, 6, 7, 8, 9}_{\text{DIGITS (10)}}\}$ $|\mathcal{A}'| = 26 + 10 = 36$

$|P'| = |P'_{10}| + |P'_{11}| + |P'_{12}| + |P'_{13}| + |P'_{14}| + |P'_{15}| + |P'_{16}| + |P'_{17}| + |P'_{18}| + |P'_{19}| + |P'_{20}|$

$= \boxed{36^{10} + 36^{11} + 36^{12} + 36^{13} + 36^{14} + 36^{15} + 36^{16} + 36^{17} + 36^{18} + 36^{19} + 36^{20}}$

$= \boxed{36^{10} + \dots + 36^{20}}$