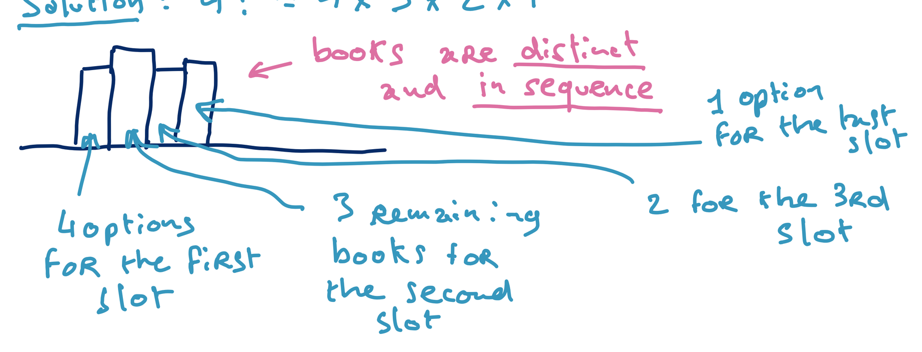
Lecture 8: More advanced counting

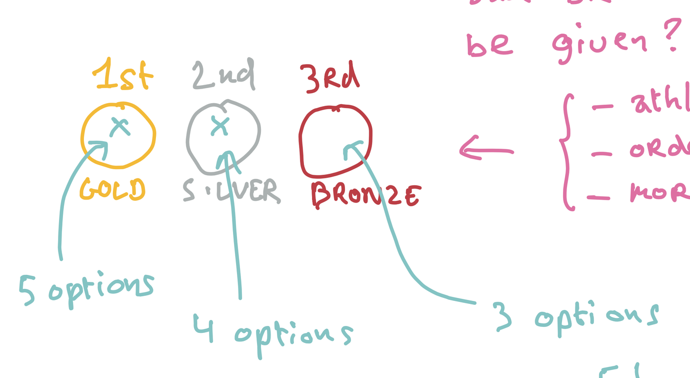
Permutation Examples

EX1. Ordering books: How many ways to order 4 different books on a shelf.

Solution: $4! = 4 \times 3 \times 2 \times 1$



EX2. Assigning medals: In a competition with 5 athletes how many ways can GOLD, SILVER and BRONZE medals be given?

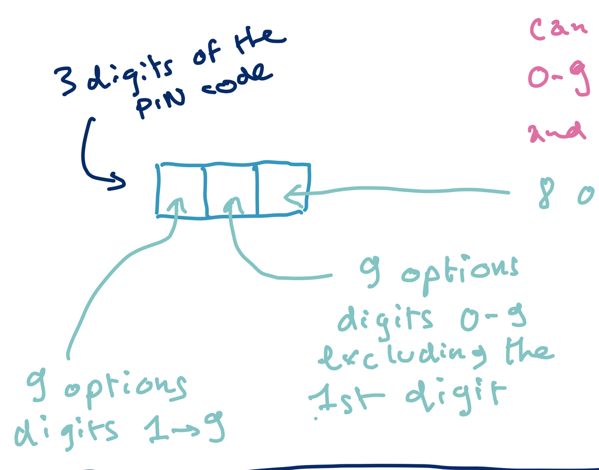


- athletes all different
- order matters
- more athletes than medals

Total ways $= 5 \times 4 \times 3 = \frac{5!}{2!} = P(5,3)$

← $(5-3)!$

EX3. Creating PIN codes: How many 3-digit PIN codes can be created using the digits 0-9 if no digit is repeated and the first digit can't be 0?



Total PIN codes $= 9 \times 9 \times 8$

COMBINATION EXAMPLES

(main difference w/ permutations is that COMBINATIONS DON'T CARE ABOUT ORDERING)

EX1. Forming Teams: From a group of 7 students, how many ways can you choose a committee of 3 students?

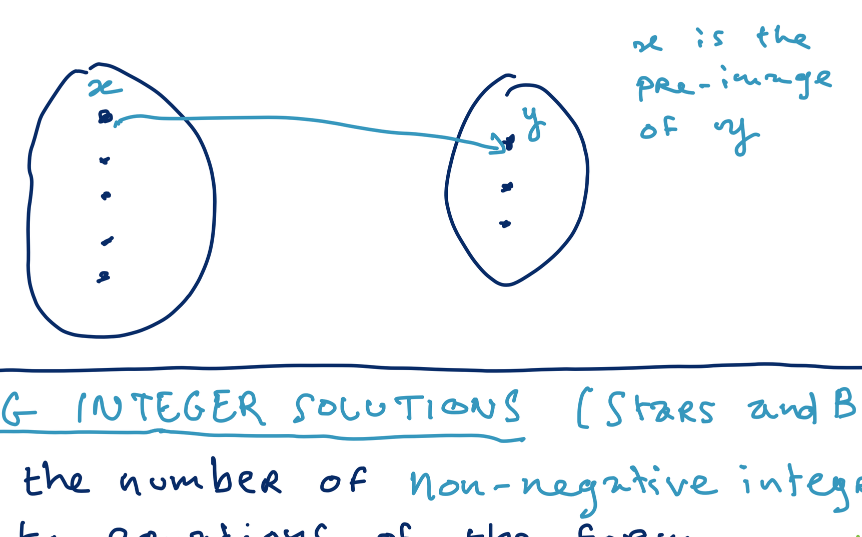
$$C(7,3) = \binom{7}{3} = \frac{7!}{3!\,(7-3)!} = \frac{7!}{3!\,4!} = \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1 \times 4 \times 3 \times 2 \times 1}$$

"7 choose 3"

$$= \frac{7 \times 6 \times 5}{3 \times 2 \times 1} = \frac{7 \times 6 \times 5}{6} = 7 \times 5 = 35.$$

EX2. Selecting Lottery Numbers. In a lottery game, you must choose 6 numbers out of 49. How many different sets of number can you draw?

$$C(49, 6) = \binom{49}{6} = \frac{49!}{6!\,(49-6)!} = \dots$$



COUNTING INTEGER SOLUTIONS (Stars and Bars)

Counting the number of non-negative integer solutions to equations of the form

$$x_1 + x_2 + x_3 + \dots + x_n = k \qquad x_i \geq 0$$

categories ($x_1, x_2, \dots$) — total units to distribute (k) — (all these are integers)

is a common problem and can be solved with the "Stars and Bars" method, which relates to combinations with repetitions.

Definition
- Stars (*) represent the units to be distributed
- Bars (|) represent the dividers between categories (x_i)

Total symbols:
- number of stars: k (units to distribute)
- number of bars: $n-1$ (dividing the units into n categories)

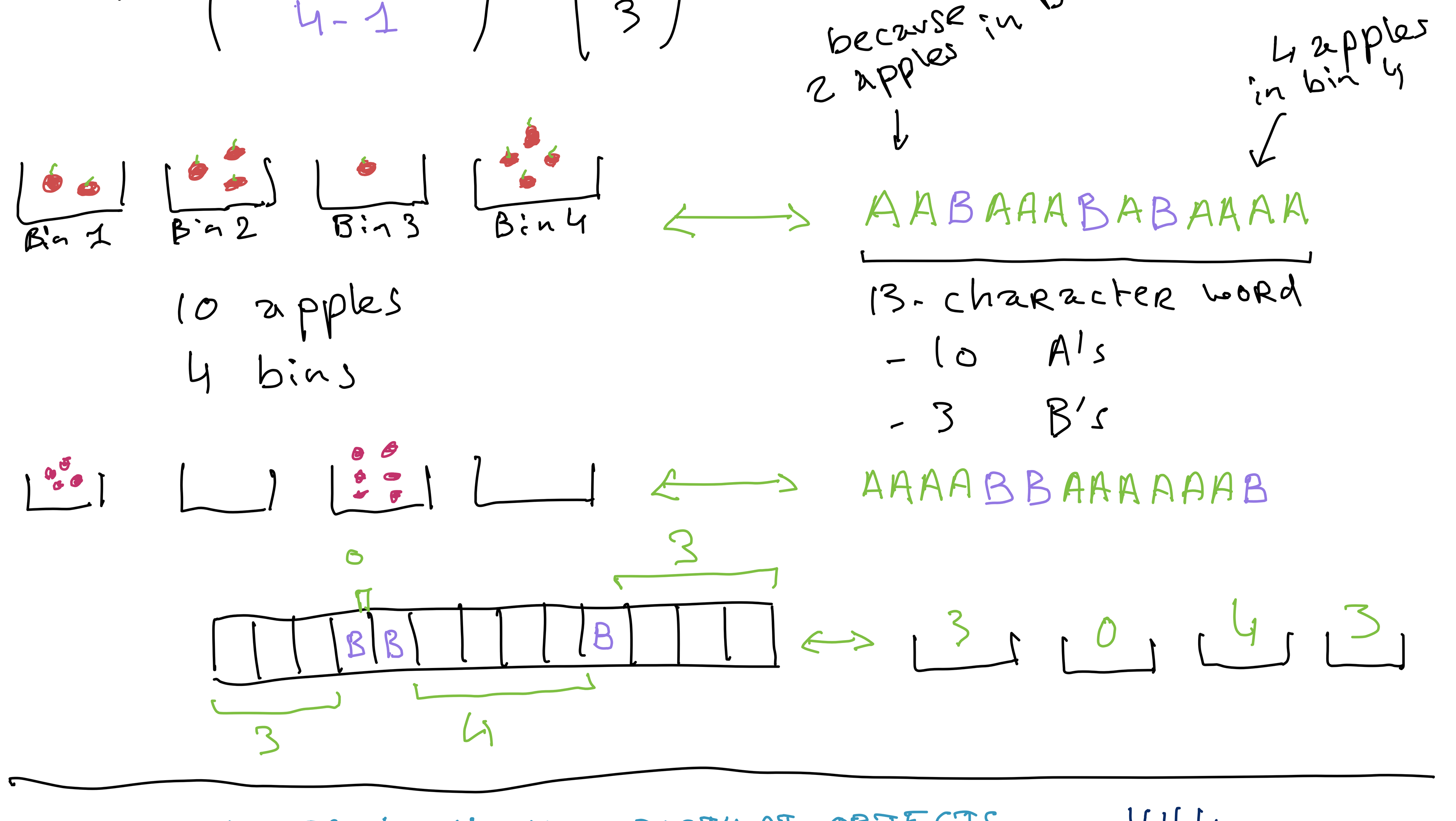
In this context, the number of total integer solutions is

$$\binom{k+n-1}{n-1}$$

EX1. How to distribute 10 apples (units to be distributed) into 4 bins (categories)?

$$x_1 + x_2 + x_3 + x_4 = 10 \qquad (x_i \geq 0)$$

$$\text{Number of ways} = \binom{10 + 4 - 1}{4 - 1} = \binom{13}{3}$$



ARRANGEMENTS WITH NON-DISTINCT OBJECTS

WILL
MISSISSIPPI

When arranging objects where some are identical the standard permutation formula $n!$ (which counts arrangements of n distinct items) overcounts the number of unique arrangements

EX. Permutations of ADD

ADD DAD DDA ADD DAD DDA

equivalent

we want to count the 3 unique arrangements as opposed to all 6.

Formula If there are n total objects with
- k_1 objects of type 1
- k_2 objects of type 2
- ...
- k_m objects of type m

and $k_1 + k_2 + \dots + k_m = n$

then the number of distinct arrangements is:

$$\frac{n!}{k_1! \times k_2! \times \dots \times k_m!}$$

EX1. How many ways to rearrange BALLOON (7 letters)

- B : 1
- A : 1
- L : 2
- O : 2
- N : 1

$$\frac{7}{1!\ 1!\ 2!\ 2!\ 1!} = \frac{7!}{2!\,2!} = \frac{7!}{4}$$