Lecture 7: Bijection Principle, Permutations and Combinations

BIJECTION PRINCIPLE

If there is a one-to-one correspondence (= bijection) between two finite sets A and B then $|A| = |B|$.

This principle is helpful when:
- counting A is "easy"
- counting B is "hard" :(
- but we can find a way to "convert" elements from A to B reversibly

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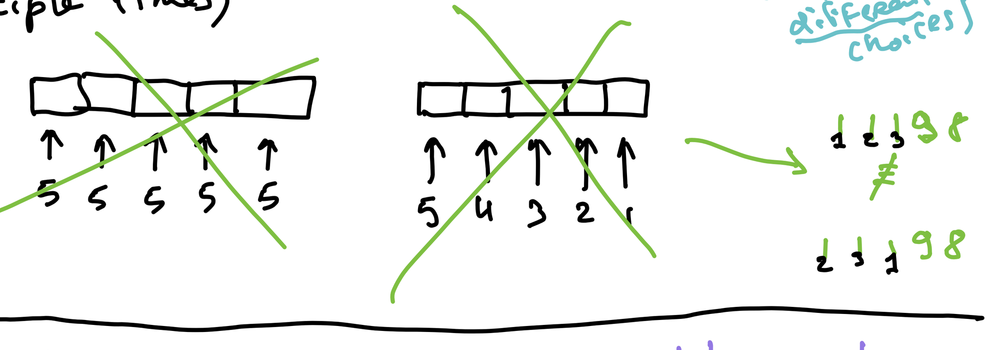
EXAMPLE 1 How many unique sequences can be formed using the digits $1_1\, 1_2\, 1_3\, 8\, 9$.

some repeated characters (previously we only had all different choices)

If we use the product rule we will overcount (= count some options multiple times)

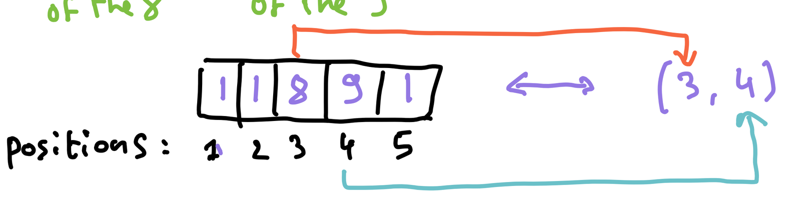


$1_1 1_2 1_3 9 8 \neq 1_2 1_3 1_1 9 8$

IDEA: instead of counting all possible numbers, try to count the ways to place the two distinct digits 8 and 9

SOLUTION:
- Set A: All sequences from 1, 1, 1, 9, 8
  11189, 11198, 18191, 19181, 11981 ...
- Set B: All ways to assign the positions of 8 and 9
  (4,5) (5,4) (2,4) (4,2) (4,3)

position of the 8 / position of the 9



$\longleftrightarrow (3, 4)$

this is the bijection: "algorithm" to convert any object from A to an object of B and reversibly

How many pairs can we form?

position of the 8 → $\boxed{5 \mid 4}$ ← pos. of 9

how many choices for 9?

how many possible choices? → we have no constraints so all 5 positions possible

$\boxed{5 \mid 4 \mid 3}$

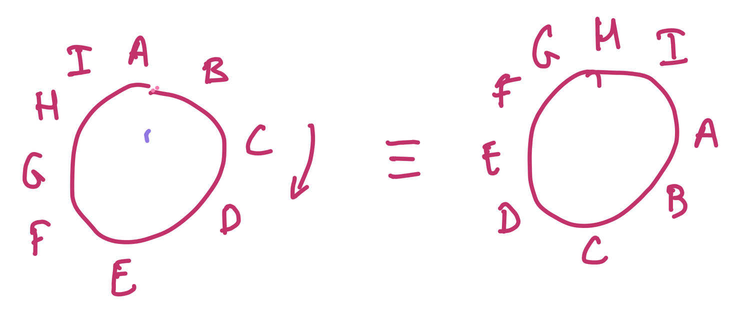
In total we have $5 \times 4 = 20$ possible pairs.

Therefore there are, by the bijection principle 20 distinct sequences of 1, 1, 1, 8, 9.

Q. Why is 8 always placed before 9?
↳ It is not physically always before 9 in the digits (eg. 19181 is possible) we just make the choices sequentially.

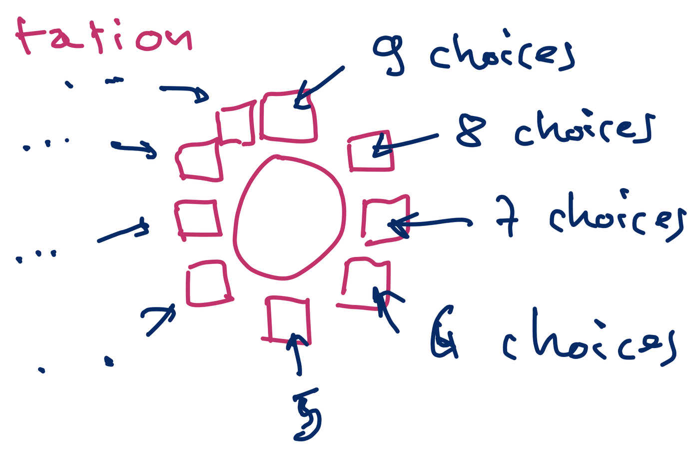
EXAMPLE 2: Circular Permutations, Seating 9 People Around a Table

Problem

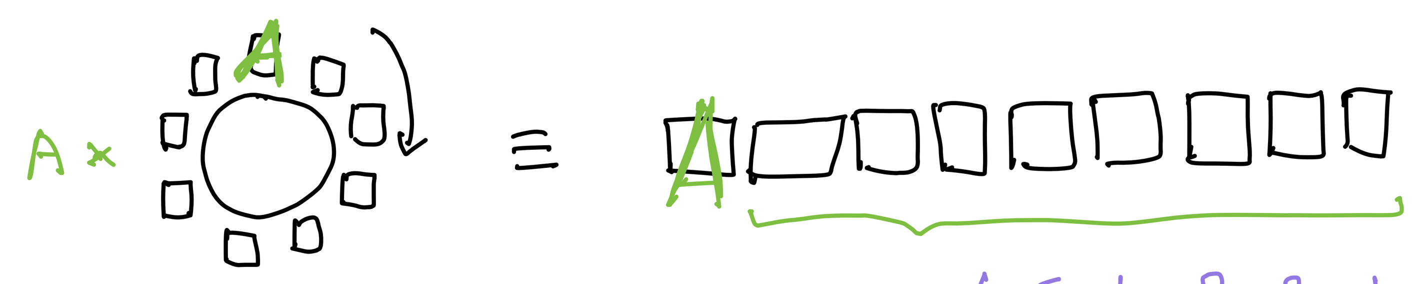


We want both these ways of setting the table to be equivalent (= counted as the same)

But if we simply try to apply the product, we will overcount because of the rotation



Solution: Fix one person in a specific seat that never changes (= breaking invariant/symmetry)



8 7 6 5 4 3 2 1 = 8!

PERMUTATIONS

A permutation is an ordered arrangement of objects.

- Formula for arranging n different objects: $n! = n \times (n-1) \times .. \times 2 \times 1$ — def: FACTORIAL
- Formula for arranging r objects from n distinct objects:

$$\boxed{\frac{n!}{(n-r)!}}$$

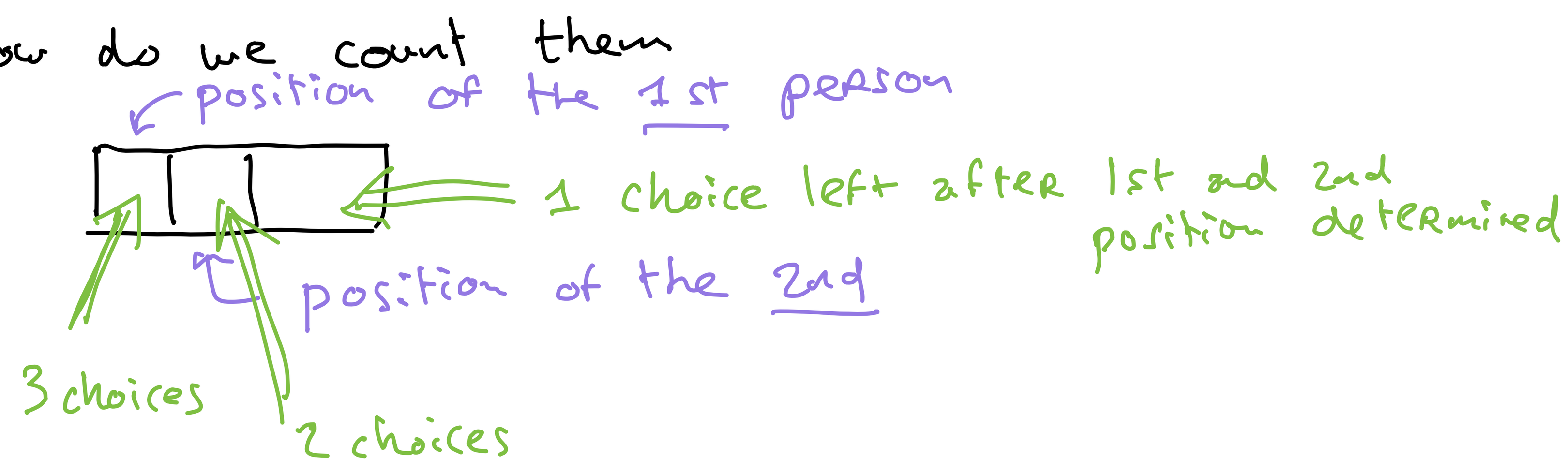
EXAMPLE 1 Arranging People in a Line

How many ways can Ash, Bo and Carmen be arranged?

A B C   A C B   B A C   B C A   C A B   C B A

↳ 6 total ways

How do we count them



$= 3 \times 2 \times 1 = 3! = 6$

COMBINATION

A combination is a selection of objects where order does not matter.

Formula: $C(n, r) = \binom{n}{r} = \frac{n!}{r!(n-r)!}$

this is read "n choose r"

this $r!$ factor is the difference between permutations and combinations

EXAMPLE 1. How many ways to choose 3 movies from a catalog of 9 movies?

$$\binom{9}{3} = \frac{9!}{3!(9-3)!} = \frac{9!}{3!\,6!} = \frac{\cancel{9}^{3} \times \cancel{8}^{4} \times 7 \times \cancel{6 \times 5 \times 4 \times 3 \times 2 \times 1}}{\cancel{3}_{1} \times \cancel{2}_{1} \times 1 \times \cancel{6 \times 5 \times 4 \times 3 \times 2 \times 1}} = 12 \times 7 = 84.$$