Lecture 6: Introduction to counting

Tentative AGENDA:
- quizzes ✓
- does reflexive mean symmetric
- why is = symmetric and anti-symmetric

then COUNTING:
- product rule
- sum rule
- bijection principle

Does reflexive mean symmetric?

endorelation

- Symmetric: a relation R on a set A is symmetric if for all $(a,b) \in A \times A$
  if aRb then bRa
- reflexive: R on A is reflexive
  if for all $a \in A$, aRa

Let's construct a relation that is symmetric but NOT reflexive.

$A = \{1, 2\}$ $\boxed{R = \{(1,2),\ (2,1)\}}$

- is R a relation? yes because $R \subset A \times A$
- is R symmetric? yes because $1R2$ implies $2R1$ and vice-versa
- is R reflexive? NO. because $\cancel{1R1}$ and $\cancel{2R2}$

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similar notations (the CS term is "syntactic sugar")
$(1,1) \in R$ is equivalent to $1R1$
$(1,1) \notin R$ $\quad 1 \not R 1$

As a follow-up the relation R' which is R extended by the two pairs $(1,1)$ and $(2,2)$:

the "missing reflexitivity pairs"

$$R' = R \cup \{(1,1),\ (2,2)\}$$

$$R' = \{(1,2), (2,1), (1,1), (2,2)\}$$

then R' is both symmetric and reflexive

COUNTING

what3words: Towne 217 is faces.lights.object

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1. PRODUCT RULE

Sometimes when counting, we are making a series of choices:
- we selecting option 1
- ______________ 2
- ...

In this situation, the product rule is useful.
The product rule comes from the rule on Cartesian Products $|A \times B| = |A| \times |B|$

little example to remember how cardinality works

$A = \{1,2\}$
$B = \{a,b\}$
$A \times B = \{(1,a), (1,b), (2,a), (2,b)\}$
$|A| = 2$
$|B| = 2$
$|A \times B| = 4 = |A| \times |B|$

Example: Ooika Matcha

Milk = { oat milk, no oat milk }

Sweet = { no sweet, light sweet, normal sweet }

How many different ways of ordering drinks?

$$|Milk \times Sweet| = |Milk| \times |Sweet| = 2 \times 3 = 6.$$

Let's add another option:

TeaType = { matcha, hojicha }

Now how many?

$$|Milk \times Sweet \times TeaType| = |Milk| \times |Sweet| \times |TeaType|$$
$$= 2 \times 3 \times 2$$
$$= 12.$$

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2. SUM RULE

If instead making choices one after another, you are counting something where you have to choose between multiple options (choice 1 OR choice 2) then the sum rule is applied

Example: The students in this classroom are divided into three groups:

Male + Female + Non-Binary = Students

because the sets are disjoint.

Example: The total number of North American citizens is the union of: U = US citizens
C = Canadian citizens
M = Mexican citizens

we can assume all North American citizens are counted by $|U| + |C| + |M|$ if we assume these sets are disjoint

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Examples using both Rules

PASSWORD CREATION EXAMPLE

How many passwords can be made using just upper and lower case letters that are between 4 and 8 characters?

formalizing the problem (this often involves defining notations)

P = set of such passwords

$$\mathcal{A} = \{\underbrace{A, B, C, D \dots Z}_{\text{UPPER CASE LETTERS}},\ \underbrace{a, b, c, \dots z}_{\text{lower case letters}}\} \quad : \text{our alphabet}$$

$|\mathcal{A}| = 26+26 = 52$

P_k = set of passwords of length k

then because the sets P_k are disjoint (elements of P_i and P_j with $i \neq j$ have different sizes)

$$P = P_4 \cup P_5 \cup P_6 \ P_7 \cup P_8$$

By the sum rule because the P_i are disjoint

$$|P| = |P_4| + |P_5| + |P_6| + |P_7| + |P_8|$$

Now how do we calculate $|P_j|$?

$$P_4 = \mathcal{A} \times \mathcal{A} \times \mathcal{A} \times \mathcal{A}$$

first letter, second letter, third letter, fourth letter

← the elements in P_4 are sequences of 4 choices of letters

By the product rule, we have

$$|P_4| = |\mathcal{A} \times \mathcal{A} \times \mathcal{A} \times \mathcal{A}| = |\mathcal{A}| \times |\mathcal{A}| \times |\mathcal{A}| \times |\mathcal{A}|$$

$$|P_4| = |\mathcal{A}|^4 = 52^4$$

Putting it all together

$$|P| = |P_4| + |P_5| + |P_6| + |P_7| + |P_8|$$

$$\boxed{|P| = 52^4 + 52^5 + 52^6 + 52^7 + 52^8}$$