CIT 5920–Mathematical Foundations of Computer Science
Homework 3: *Advanced counting*

Version: September 24, 2024

Complete the online portion on PrairieLearn. For the written section, use the HW1 template on Overleaf, shared on the course forum and Canvas. Each exercise should be on a separate page. Only submissions in this format will be accepted. Submit your written work on Gradescope by the deadline. For assistance or inquiries, don't hesitate to: Attend office hours; post questions on the class forum; ask about the motivation behind this material.

Guidelines:

- Ensure clarity in your answers.
- Avoid direct answers unless specified. Merely providing a number will result in deductions.
- Clearly state any assumptions in combinatorics questions. Reasonable assumptions will be credited. For instance, assuming two people are identical is not reasonable, neither is assuming that the order matters when mixing paints.
- Prefer math symbols over words, e.g., use $A \cap B$ instead of "The elements common to both A and B."
- If unsure about the length or style of your answer, consult during office hours or post on the class forum. However, we won't provide direct answers.
- For the PrairieLearn section, consider using Wolfram Alpha's Equation Solver. For the written section, avoid calculators. Answers can be in factorial or $\binom{n}{k}$ form.

Exercise 0 – PrairieLearn Questions [14pts]

Use the QR code to the right to access the PrairieLearn portion of this homework. **Please login using your Penn Google account.**

1. Most questions are designed to provide you with an infinite number of variations.
2. With these questions, if you respond incorrectly, you will have the opportunity to try again until you get the question right. To earn credit on the question, you must answer *any* variant from the first try.

Exercise 1 – Arrangement and Grouping of Books on Shelves [7pts]

The Penn book store has 10 copies (identical) of a Python textbook, 6 (identical) copies of a mathematics textbook and just 4 (identical) copies of a Java programming textbook. These textbooks are going to all be arranged. In how many ways can they all be arranged if:

A. They have to go on a single shelf and you can have them in any order?

B. They have to go on a single shelf and you are trying to be sensible and put the Python books together, the math books together and the Java books together?

C. There are now 4 distinct shelves. You want to have a shelf of Java, a shelf of Python and a shelf of mathematics (leaving one shelf empty). In how many ways can this be achieved?

D. You again have 4 distinct shelves, but you can have any sort of arrangement in those shelves (including having or not empty shelves). In how many ways can this be achieved?

Exercise 2 – Circular Permutations with a Constraint [4pts]

We suggest you attempt solving the "Circular Seating with Constraints" problem (HW3.8) in the online part first, before attempting this problem. At a wedding reception, there are 10 round tables numbered 1 through 10. Each round table seats 12. There are 60 couples at this wedding.

In how many ways can these couples be seated if there are no constraints other than the fact that a couple has to be seated at the same table (but not necessarily with both members of the couple seated next to each other)?

Exercise 3 – Subset Triplets and Formal Reasoning [4pts]

Given a set W of size k, in *how many ways* can we make ordered triplets of sets (A, B, C) which satisfy the conditions:

- A, B, C are all subsets of the original set
- $A \subseteq B \subseteq C$ (recall that $\subseteq$ is the "subset or equal" relation).

For instance, in the case when the set is $\{1, 2\}$ we have the following possibilities

- $(\emptyset, \emptyset, \emptyset)$
- $(\emptyset, \emptyset, \{1\})$
- $(\emptyset, \emptyset, \{2\})$
- $(\emptyset, \emptyset, \{1, 2\})$
- $(\emptyset, \{1\}, \{1\})$
- $\ldots$
- and some more...

How many such ordered triplets can we form from a set W of size k?

Exercise 4 – Counting Relations and Formal Reasoning [2+3pts]

In this exercise, we are going to count the number of reflexive relations on a set A of size n, $n > 0$. Recall that a relation on a set A is a subset of the Cartesian product $A \times A$; any subset from $\emptyset$ to $A \times A$ is a relation. A reflexive relation R on set A is one in which *all* the pairs (x, x) for x in A belong to the relation R.

A. How many reflexive relations are there on the set $\{1, 2, 3, 4\}$?

B. Based on this, what is the general formula, of the number of reflexive relations, for an n element set where $n > 0$?