CIT 5920
Recitation 2

Sep. 13, 2024 — Fall 2024

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Overview for Today

- Logistics (5 minutes)
- Review (45 minutes)
- Question 1 (3 minutes)
- Question 2 (4 minutes)
- Question 3 (4 minutes)
- Question 4 (3 minutes)
- Question 5 (5 minutes)
- Homework

Logistics

- TA Office Hours starting next week
- Homework 1 released
  - Due Monday, September 16 at 11:59 PM, but we will accept late submissions without penalty
  - Homework 2 will be released next Monday (due Monday, September 23 at 11:59 PM)
- Math Resources

Review

- (Generalized) Cartesian product
- A_i notation
- Partitioning sets
- Practice distinguishing element/subset of A, B and A × B
  - A and B contain elements and A × B contains pairs [of elements]

Review

Generalized Cartesian products

The Cartesian product of $A_1, A_2, \ldots, A_n$, denoted $A_1 \times A_2 \times \ldots \times A_n$, is defined to be:

$A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) : a_i \in A_i \text{ for all integers } i \text{ such that } 1 \leq i \leq n\}$

What is the cardinality of the cross product of a set with cardinality m and a set with cardinality n ?

Review

Partition

A set *A* can be partitioned into subsets A_i if

$$A = \bigcup_{i=1}^{n} A_i \text{ and}$$

$$A_i \cap A_j = \emptyset \text{ for every } i \neq j.$$

Review

$A = \{1, 2\}$ $B = \{5, 1\}$

What is $A \times B$?

Review

$A = \{1, 2\}$ $B = \{5, 1\}$

What is $A \times B$?

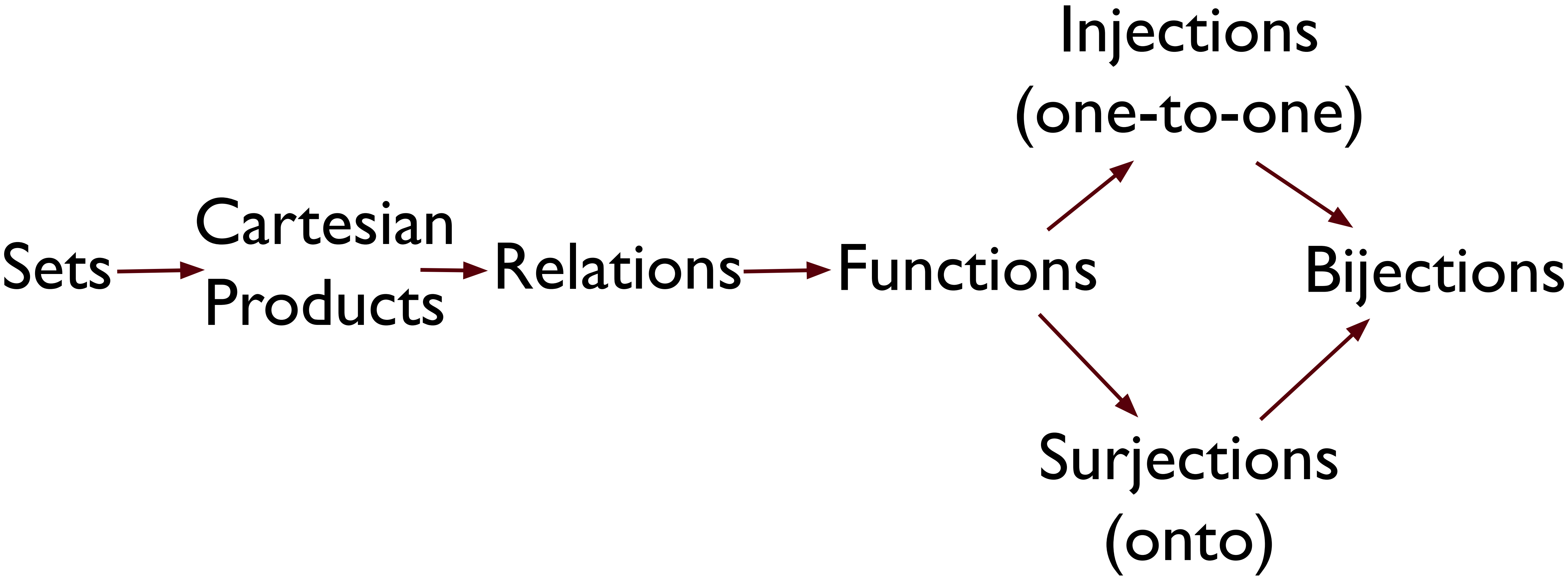
Is $\{1, 5\} \subseteq A \times B$?

Is $\{1, 5\} \in A \times B$?

Is $\{(1, 5)\} \subseteq A \times B$?

Is $\{(1, 5)\} \in A \times B$?

Roadmap


Sets
Cartesian Products
Relations
Functions
Injections (one-to-one)
Surjections (onto)
Bijections

Review

- Relations are subsets of Cartesian products
  - Reflexive: if **for all** elements *a* in set *A*, *aRa*.
  - Symmetric: if *aRb* means that *bRa*.
  - Transitive: if whenever *aRb* and *bRc*, then *aRc*.
  - Equivalent: iff (if and only if) reflexive, symmetric, and transitive.
  - Anti-symmetric:
    1. if *aRb* and *bRa*, then $a=b$.
    2. if *aRb* with $a \neq b$, then *bRa* must not hold.
- Examples

Review

Example: *aRb* iff User a follows User b on X (Twitter)

Is this reflexive? (Do we need to assume anything?) Y?

Is this symmetric? N

Is this transitive? N

Is this an equivalence relation? N

Is this anti-symmetric? N

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Review

Example: *aRb* iff *a* is a subset of *b*

Is this reflexive?	Y
Is this symmetric?	N
Is this transitive?	Y
Is this an equivalence relation?	N
Is this anti-symmetric?	Y

Review

Example: *aRb* iff Person a lives within 2 miles of Person b

Is this reflexive?	Y
Is this symmetric?	Y
Is this transitive?	N
Is this an equivalence relation?	N
Is this anti-symmetric?	N

Review

Is it possible for a relation to be both symmetric and anti-symmetric?

a = b

Review

If a,b $\in$ R, determine whether these relations below are reflexive, symmetric, transitive, and anti-symmetric

- $a \leq b$
- a != b
- a^2 = b^2

Solution

Relation	Reflexive	Symmetric	Transitive	Anti-symmetric
$a \leq b$	Yes	No	Yes	Yes
$a \neq b$	No	Yes	No	No

Solution

3. Relation: $a^2 = b^2$ (Equality of Squares)

- **Reflexive**:
  Yes, the relation is reflexive because for all real numbers a, $a^2 = a^2$ holds. This satisfies the definition of reflexivity.
- **Symmetric**:
  Yes, the relation is symmetric. If $a^2 = b^2$, then $b^2 = a^2$, which holds true by the equality of the squares.
- **Transitive**:
  Yes, the relation is transitive. Transitivity requires that if $a^2 = b^2$ and $b^2 = c^2$, then $a^2 = c^2$. This holds true because if the squares of a, b, and c are equal, then $a^2 = c^2$.
- **Anti-symmetric**:
  No, the relation is not anti-symmetric. For anti-symmetry, if $a^2 = b^2$ and $b^2 = a^2$, then it should imply that $a = b$. However, this is not the case here because $a^2 = b^2$ can be true even when $a \neq b$. For example, $a = 1$ and $b = -1$ have equal squares but are different values.

Review

Let A={a,b,c,d}. Consider the Cartesian product A×A which includes all possible ordered pairs from elements of A and form these relations below from the set A×A

EX: Reflexive $R1 = \{(a,a),(b,b),(c,c),(d,d)\} \cup (\text{any other pairs from } A \times A)$

- Transitive
- Anti-symmetric
- Reflexive, symmetric, transitive, and anti-symmetric

Solution

Transitive:

$$R3 = \{(a,a),(b,b),(c,c),(d,d),(a,b),(b,c),(a,c),(a,d),(b,d),(c,d)\}$$

Anti-symmetric

$$R4 = \{(a,a),(b,b),(c,c),(d,d),(a,b),(b,c),(c,d)\}$$

Reflexive, symmetric, transitive, and anti-symmetric

$$\{(a,a),(b,b),(c,c),(d,d)\}$$

Question 1

Assume A and B are sets.

Under what conditions is $A \times B = B \times A$?

In arithmetic you are told that if $a - b = 0$ then $a = b$.

In set theory, if someone tells you $A - B = \emptyset$ what will that mean?

Answer 1

Either A = B, or at least one set is empty

(Since (a, b) are ordered pairs, the only way the pairs we build are the same from AxB and BxA is if: EITHER the Cartesian products are the same (since A is identical to B) or the Cartesian product is empty (if one of A or B is empty))

$$A \subseteq B \text{ (B contains all elements of A)}$$

Question 2

For each of the following, figure out if they are Reflexive, Symmetric, Transitive, and Anti-symmetric. Assume that in each case, the relation that we are talking about is named R and when we say aRb, we mean a is related to b, that is $(a, b) \in R$.

(a) On the set of real numbers, a is related to b iff $|a - b| < 5$.

Answer 2a

On the set of real numbers, a is related to b iff $|a - b| < 5$. **Solution:**

Reflexive: Yes because every number subtracted from itself is 0, which is less than 5.
Symmetric: Yes because $|a - b| = |b - a|$ so if $|a - b| < 5$, then $|b - a| < 5$
Transitive: No. $(5, 1)$ and $(1, -1)$ are in the relation but $(5, -1)$ is not
Anti-Symmetric: No. $(3, 2)$ and $(2, 3)$ are in the relation but $3 \neq 2$

Question 2

For each of the following, figure out if they are Reflexive, Symmetric, Transitive, and Anti-symmetric. Assume that in each case, the relation that we are talking about is named R and when we say aRb, we mean a is related to b, that is $(a, b) \in R$.

(b) Assume the set A is partitioned into subsets $A_1, A_2, \ldots, A_n$. Define the relation R between elements of A as aRb if a and b are elements of the same A_i.

(c) Consider the set of human beings. Define two humans being related by R_b as $h_1 R_b h_2$ if and only if h_1 is a biological child of h_2.

(d) On $\mathbb{N}$, aRb iff a divides b. divides in math is defined as follows

a divides b if there is some(at least one) integer solution to the equation $ax = b$

Answer 2b

Assume the set A is partitioned into subsets $A_1, A_2, \ldots, A_n$. Define the relation R between elements of A as aRb if a and b are elements of the same A_i. **Solution:**

Reflexive: Yes because each element of A_i is in the same subset as itself.
Symmetric: Yes because if a is in the same subset A_i as b, then b is in the same subset A_i as a.
Transitive: Yes because if a is in the same subset A_i as b and b is in the same subset A_i as c, then a and c are also in the same subset A_i.
Anti-Symmetric: No because if a can be in the same subset A_i as b and b is in the same subset A_i as a, but $a \neq b$.

Answer 2c

Consider the set of human beings. Define two humans being related by R_b as $h_1 R_b h_2$ if and only if h_1 is a biological child of h_2. **Solution:**

Reflexive: No because a child cannot be their own parent.
Symmetric: No because child of a parent cannot be a parent of their parent.
Transitive: No because a child of a parent is not a child of their grandparent.
Anti-Symmetric: Yes because there isn't a relationship where a child is also the parent of their parent in this relation; therefore, this is vacuously true.

Answer 2d

On $\mathbb{N}$, aRb iff a divides b. divides in math is defined as follows a divides b if there is some(at least one) integer solution to the equation $ax = b$ **Solution:**

Reflexive: Yes because a number divided by itself is 1.
Symmetric: No. $(2, 1)$ are in the relation but $(1, 2)$ are not.
Transitive: Yes. Let $a = bx$ and $b = cy$. By substitution, $a = cxy$. Therefore, by definition, c divides a. In other words, if a is divisible by b and b is divisible by c, then a is divisible by c.
Anti-Symmetric: Yes because if a is divisible by b and b is divisible by a, this implies that $a = b$
Note: The definition provided for division allows integers to be divided by 0

Question 3

True or False. A relation has to be either reflexive or symmetric or transitive or anti-symmetric. In other words, every relation must satisfy at least one of the 4 properties. If you think this True write out a justification (the concept of a formal proof will be covered later). If you think this is False then write down a non-empty set and a non-empty relation on that set that does not satisfy any of these 4 properties.

Forcing yourself to find/create/generate examples is one of the biggest muscles this course tries to exercise!

Answer 3

This statement is **false**. Consider set $\{a, b, c, d\}$ and the relation $\{(a, b), (b, c), (c, a), (a, c)\}$. It is not reflexive (no element paired with itself), not symmetric ((a,b) but no (b,a), not transitive ((b,c) and (c,a) but no (b,a), and not antisymmetric ((a,c) and (c,a) included but $c \neq a$).

This is just one example and there are many other possibilities!

If you found a different one, it is even better! Don't hesitate to ask TAs to check your example to be sure.

Question 4

Set builder notation review

Write the following sets in set builder notation:

a) A = {0.25, 0.5, 1, 2, 4, 8, 16, 32}
b) B = {1, 4, 9, 16, 25, 36}
c) C = {a, c, e, h, i, m, s, t}

Question 4a

A = {0.25, 0.5, 1, 2, 4, 8, 16, 32} can be written as:

$\{x \mid x = 2^n, n \in \mathbb{Z}, -2 \leq n \leq 5\}$

Question 4b

B = {1, 4, 9, 16, 25, 36} can be written as:

$\{x \mid x \in \mathbb{Z}, x = n^2, 1 \leq n \leq 6\}$ or

$\{x \mid x = n^2, x \in \mathbb{Z}, 1 \leq n \leq 6\}$

$\{x \mid x \in \mathbb{Z}, x = n^2, -6 \leq n \leq 6\}$ or

$\{x \mid x = n^2, x \in \mathbb{N}, n \in \mathbb{N}, 1 \leq x \leq 36\}$

$\{x \mid x \in \mathbb{N}, \sqrt{x} \in \mathbb{N}, 1 \leq x \leq 36\}$ and so on and so on

We can bound x. We can bound n. But this may be redundant.

All these are correct. But try be as accurate as possible

Question 4c

C = {a, c, e, h, i, m, s, t} can be written as:

{x | x ∈ 'mathematics'}

Question 5

Prove it by the set identities $(A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A)$

Set Identities

Name	Identities	
Idempotent laws	$A \cup A = A$	$A \cap A = A$
Associative laws	$(A \cup B) \cup C = A \cup (B \cup C)$	$(A \cap B) \cap C = A \cap (B \cap C)$
Commutative laws	$A \cup B = B \cup A$	$A \cap B = B \cap A$
Distributive laws	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Identity laws	$A \cup \emptyset = A$	$A \cap U = A$
Domination laws	$A \cap \emptyset = \emptyset$	$A \cup U = U$
Double complement law	$\overline{\overline{A}} = A$	
Complement laws	$A \cap \overline{A} = \emptyset$ $\overline{U} = \emptyset$	$A \cup \overline{A} = U$ $\overline{\emptyset} = U$
De Morgan's laws	$\overline{A \cup B} = \overline{A} \cap \overline{B}$	$\overline{A \cap B} = \overline{A} \cup \overline{B}$
Absorption laws	$A \cup (A \cap B) = A$	$A \cap (A \cup B) = A$

definition of set difference, $X \setminus Y = X \cap Y^c$, where Y^c is the complement of Y.

Solution

LHS = $(A \cup B) \cap (\text{not}(A \cup B))$ [set difference]

$= (A \cup B) \cap (\text{not}A \cup \text{not}B)$ [De Morgan's law]

$= (A \cap \text{not}A) \cup (A \cap \text{not}B) \cup (B \cap \text{not}A) \cup (B \cap \text{not}B)$ [Distributive]

$= (A \cap \text{not}B) \cup (B \cap \text{not}A)$

$= (A \setminus B) \cup (B \setminus A)$ = RHS

See you next week!

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