Lecture 7: Functions

Sets $\xrightarrow{\text{make pairs}}$ Cartesian Products $\xrightarrow{\text{subsets}}$ Relations ⇢ Functions ???

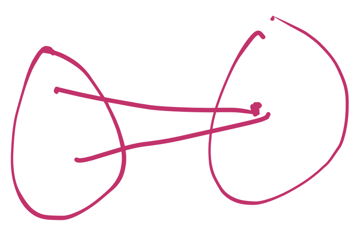
A function f from set X to set Y, denoted

$f: X \to Y$

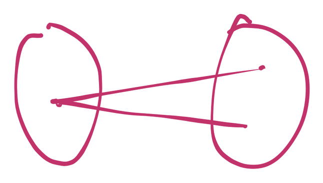
is a relation with the property that

Every element in X is related to exactly one element in Y

This means
- Totality: For every $x \in X$, there exists $y \in Y$, $(x, y) \in f$
- Uniqueness: For every $x \in X$, there is exactly one $y \in Y$, $(x,y) \in f$



OK for a function



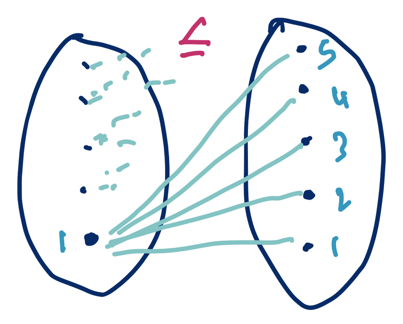
NOT OK for a function

$\text{Sqrt}: \mathbb{R}^+ \to \mathbb{R}^+$

Terminology
- X: domain, input [set], source
- Y: co-domain, output [set], range, target

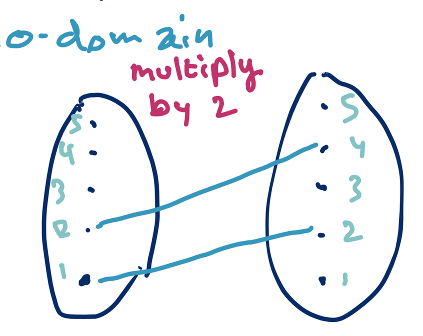
↳ can also mean only the values reached by the function

⚠ Common confusion
- All functions are relations, but not all relations are functions
- In a function, an element in the domain, cannot be related to more than one element in the co-domain



(only domain relation of 1 here)

relation because more than one element in relation with 1



multiply by 2

function because the elements on the left have exactly one element they are in relation with

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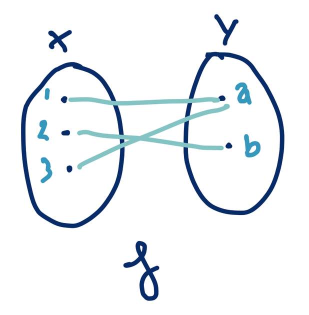
Example 1: let $f: \mathbb{N} \to \mathbb{N}$ be defined as $f(n) = n^2$
- function: for each natural number n, $f(n)$ is uniquely defined
- domain: $\mathbb{N}$ ← needs to associate to each element
- co-domain: $\mathbb{N}$ ← it's okay if not all numbers of this can be reached by the function

Example 2: let $X = \{1,2,3\}$ $Y = \{a, b\}$ and define f as:

$f(1) = a$ $\qquad f(2) = b$ $\qquad f(3) = a$

~~$f(1) = b$~~



Example 3: consider relation $R = \{(1,a), (1,b), (2,b)\}$ from $X = \{1,2\}$ to $Y = \{a,b\}$
- Issue: 1 is related to both a and b
- Conclusion: This is not a function because it violates uniqueness

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Properties of Functions

INJECTIVE ("one-one" or "one-to-one")

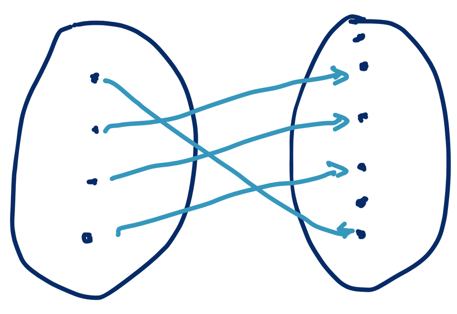
SURJECTIVE

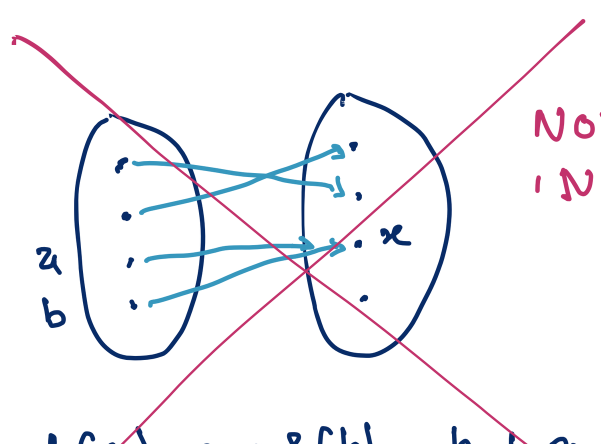
BIJECTIVE

INJECTIVE FUNCTION ("one-one")

A function $f(x)$ is injective if and only if

$f(a) = f(b)$ implies $a = b$





NOT INJECTIVE

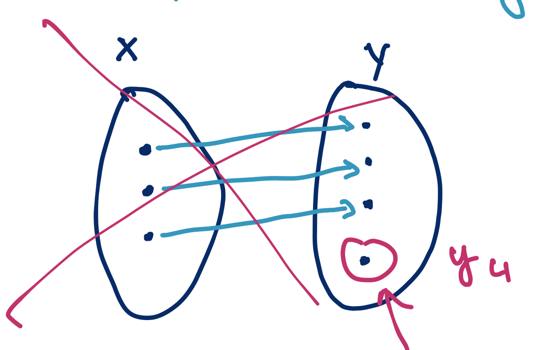
$f(a) = x = f(b)$ but $a \neq b$

SURJECTIVE FUNCTION ("onto")

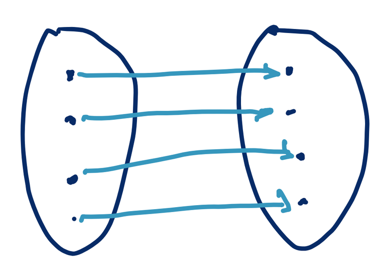
A function $f: X \to Y$ is said to be surjective if and only if the entire co-domain is covered (/reached)

if for every $y \in Y$ there exists $x \in X$

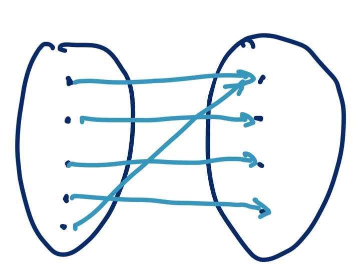
such that $f(x) = y$



A NOT SURJECTIVE — not reached, no x such that $f(x) = y_4$



B YES: all elements are reached



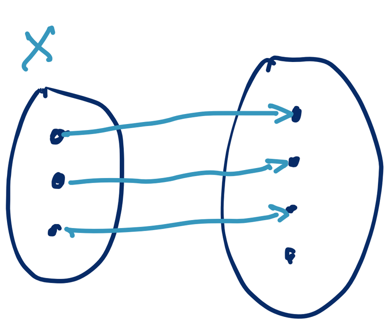
C YES: all elements are reached

if $|X| < |Y|$

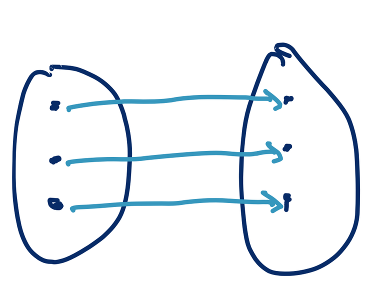
the function CANNOT BE SURJECTIVE

BIJECTIVE FUNCTION

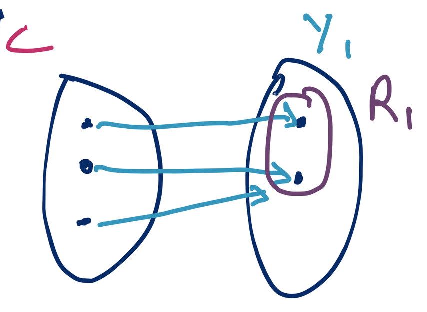
A function $f: X \to Y$ is bijective if and only if it is BOTH: INJECTIVE and SURJECTIVE



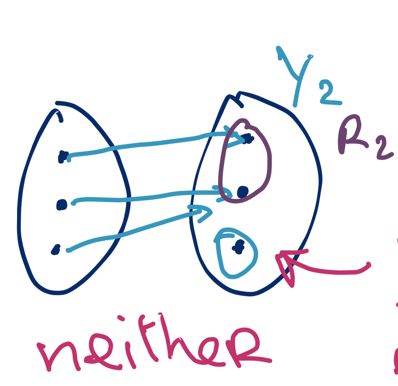
a INJECTIVE



b BIJECTIVE



c SURJECTIVE



d neither

$R_1 = R_2$

$Y_1 \neq Y_2$

we added an extra element compared c) and now not surjective

QUIZ