Lecture 6: Properties of Relations

$A = \{a, b\}$ $B = \{1, 2, 3\}$

1. What is $A \times B$?

$A \times B = \{ {}^{1}(a,1), {}^{2}(a,2), {}^{3}(a,3), {}^{4}(b,1), {}^{5}(b,2), {}^{6}(b,3) \}$

$|A \times B| = |A| \times |B| = 2 \times 3 = 6$

2. How many subsets are there for $A \times B$ $|\mathcal{P}(A \times B)|$? $= 2^{|A \times B|}$

↑ the 2 is "number of options for each element"
→ opt 1 include in subset
→ opt 2 exclude

A relation R over a set A, so $R \subseteq A \times A$ is said to be (the starting and destination set are the same)

relation from A to B ← start ← end

- **REFLEXIVE** if and only if
  $(x, x) \in R$ for all $x \in A$
- **SYMMETRIC** if and only if (this notation is more compact)
  $a R b$ means we also have $b R a$ ⎫ equivalent notations
  $(a,b) \in R$ $(b,a) \in R$ ⎭
- **TRANSITIVE** if and only if
  $a R b$ and $b R c$ means $a R c$
  $(a,b) \in R$ $(b,c) \in R$ $(a,c) \in R$
  [example: $>$ (larger than): $3 > 2$, $2 > 1$, $3 > 1$]
- **ANTI-SYMMETRIC** if and only if
  $a R b$ and $b R a$ imply $a = b$
  [example: $\leq$ (lesser or equal to)]
- **EQUIVALENCE RELATION** if and only if it is:
  - Reflexive
  - symmetric
  - transitive

  [example: $=$ (equal)]

EXAMPLES of RELATIONS → $R \subseteq A \times A$

- $A = \{1, 2, 4\}$ and let R to be the relation $\leq$ (less or equal to) that is $a R b$ if and only if $a \leq b$.
  - is this reflexive? YES, for all $x \in A$, $x \leq x$, so $x R x$.
  - is this symmetric? NO, counter-example: $1 \leq 4$ but ~~$4 \leq 1$~~
  - is this transitive? YES, if $(a,b) \in R$ then $a \leq b$
    if $(b,c) \in R$ then $b \leq c$
    but then (because $\leq$ is transitive) we have $a \leq c$
    therefore $(a,c) \in R$ (or $a R c$)
  - is this equivalence relation? NO since not symmetric
  - is this anti-symmetric? YES because $a \leq b$ and $b \leq a$ implies $a = b$.
- Two examples of relations (Mia's question)
  - Relation on $A \times B$ ← courses (↑ students)
    - courses that students are taking
    - courses that students failed
    - courses that students TA

    { (Mia, CITS9220)
    (Joe, CITS9330)
    (Iso, CITS9220)
    (Iso, CITS9330)
    ... }
  - Relation on $A \times A$ ← students (↑ students)
    - is older than
    - is in love with
    - is working with
    - is angry with

    { (Mia, Iso),
    (Iso, Iso)
    (Mia, Mia)
    (Iso, Mia)
    (Fe, Iso)
    ... }

    this is necessary to talk about -symmetric -reflexive

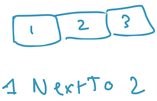
LAST EXAMPLE OF RELATIONS

$U = \{$ all states in the USA $\}$

Relation NextTo = relation if two states are geographically contiguous (share a border)

From notation: $(NJ, PA) \in$ NextTo NJ NextTo PA

- reflexive? inconclusive
- symmetric? YES (ie NJ NextTo PA implies PA NextTo NJ)
- transitive? NO (counter-example = one rotten apples)



1 NextTo 2
2 NextTo 3
but ~~1 NextTo 3~~

- equivalence? No
- anti-symmetric? No
  (NJ, PA) $\in$ NextTo
  (PA, NJ) $\in$ NextTo
  NJ $\neq$ PA