Lecture 4: Relations & Functions

Functions

Definition of a relation

A relation between set A and set B is a subset of $A \times B$. While for the purposes of pure math there does not need to be any underlying property that governs the relation, for most practical purposes, you will find that there will be some property.

For example: Consider the set $A = \{1,2,3\}$ and $B = \{4,5,6\}$ and let us define the relation R to consist of tuples $\{(a,b) | a \in A, b \in B \text{ and } b = a+3\}$. Then the relation $R = \{(1,4),(2,5),(3,6)\}$.

If $(a,b) \in R$, then this is very often denoted as aRb. That notation is similar to the way we write out relations like greater than, less than etc.

Properties

- A relation R on set A is reflexive if **for all** elements a in set A, aRa.
- A relation R from set A to set B is said to be symmetric if aRb means that bRa (implicitly we are assuming $a \in A$, $b \in B$).
- A relation R on a set A is said to be transitive if whenever aRb and bRc then aRc.
- A relation R is said to be an equivalence relation if and only if it is reflexive, symmetric and transitive.
- A relation R is said to be anti-symmetric if aRb and bRa can only happen when $a = b$.

  Alternatively we can define anti-symmetric as $x \neq y$ should imply that either x is not related to y or y is not related to x.

Examples

1. Given the set $A = \{1,2,4\}$, define the relation R to be the $\leq$ relation. That is aRb whenever $a \leq b$.

   Is this reflexive? Yes. For $a \leq a$ is always true. In particular, it is true for elements of A.

   Is this symmetric? No. For instance $1 \leq 2$ but it is not the case that $2 \leq 1$.

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Notions coming up:

- relations
- properties of relations

these are built using sets

- functions (a special kind of relation)
- properties of functions
  - injection
  - surjection
  - bijection

$$\boxed{\text{A relation between set } A \text{ and set } B \text{ is a subset of } A \times B.}$$

Example: $A = \{1,2,3\}$ $B = \{a,b,c\}$

$A \times B = \{(1,a), (2,a), (3,a), (1,b), (2,b), (3,b), (1,c), (2,c), (3,c)\}$

relations

$X_1 = \{(1,a), (2,b), (3,c)\}$

$X_2 = \{(2,a)\}$

$X_3 = \{\}$

$X_4 = \{(3,b), (3,c)\}$

any subset of $A \times B$ is a relation from A to B

"relation between A and B"

Notations: if X_1 is a relation then it is equivalent to write $(1,a) \in X_1$ and $1 \; X_1 \; a$

the relation can be used as an "operator"

PROPERTIES of RELATIONS

A relation R

- is said to be reflexive over set A if and only if $(x,x) \in R$ for all $x \in A$
- is said to be symmetric if and only if aRb means that bRa
- is said to be transitive if and only if aRb and bRc means aRc

In addition, we have:

- R is an equivalence relation if and only if it is reflexive, symmetric, transitive.
- R is anti-symmetric if and only if aRb and bRa imply $a=b$

not to be confused with ASYMMETRIC which is the direct opposite of SYMMETRIC

(but a relation CAN BE BOTH symmetric and anti-symmetric)

e.g. $X = \{(a,a), (b,b)\}$

EXAMPLES of RELATIONS

- $A = \{1,2,4\}$ and define R to be the $\leq$ relation (aRb means $a \leq b$)
  - is this reflexive? yes, because for all $x \in A$, $x \leq x$ (xRx)
  - is this symmetric? no, because $1 R 4$ (because $1 \leq 4$) ($(1,4) \in R$) but $4 \cancel{R} 1$ (because $\cancel{4 \leq 1}$ is not true) ($(4,1) \notin R$)
  - is this transitive? yes, if $(a,b) \in R$ and $(b,c) \in R$ then $a \leq b$ and $b \leq c$, therefore $a \leq c$, so $(a,c) \in R$
  - is this anti-symmetric? ...