Lecture 4:

Detail on previous lecture

A is distributive over B
if: $A(x, B(y,z)) = B(A(x,y), A(x,z))$ — postfix notation (operator first)

$x\ A\ (y\ B\ z) = (x\ A\ y)\ B\ (x\ A\ z)$ — infix notation (operator in middle)

operators: A, B

↳ union is distributive intersection

$\cup(x, \cap(y,z)) = \cap(\cup(x,y), \cup(x,z))$

intersection is distributive on union

$\cap(x, \cup(y,z)) = \cup(\cap(x,y), \cap(x,z))$

$5^{x+y} \neq 5^x \times 5^y$

$= 5^{x \cdot y}$

two operations: - exponent - multiplication } exponent is not distributive over multiplication

$a + (b \times c) \neq (a+b) \times (a+c)$

addition is NOT DISTRIBUTIVE over product

$a \times (b+c) = (a \times b) + (a \times c)$

product is DISTRIBUTIVE over addition

More definitions

CARDINALITY: the size (= number of unique elements) of a set S is notated

$|S|$ or $card(S)$

e.g. $|\{1,2,5\}| = 3$ $\quad |\mathbb{N}| = \infty$

$card(\{1,2,5\}) + card(\{1,2,2,5\}) = 3$

⚠ repetitions DO NOT matter

$|\{x\}| = |\{x, x, x, x, x, x, x \ldots\}| = 1$

e.g. $card(\{\}) = card(\emptyset) = 0$

$card(\{\{\}\}) = 1$

$card(\mathbb{Z}) = card(\mathbb{Q}) = card(\mathbb{R}) = \infty$

(these special sets all have infinite cardinality, but increasing density)

POWERSET given a set S, the powerset $\mathcal{P}(S)$ is the set of all subsets of S (or equal)

e.g.: $X = \{3, 9\}$

$\mathcal{P}(X) = \{\emptyset, \{3\}, \{9\}, \{3,9\}\}$

$|\mathcal{P}(X)| = 4$

$\mathcal{P}(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$

e.g.: $\mathcal{P}(\emptyset) = \{\emptyset\}$ $\quad card(\mathcal{P}(\emptyset)) = 1$

⚠
- $\emptyset$ is always part of the powerset
- S (the set itself) is also always part of the powerset

e.g.: $X = \{a, b, \{3\}\}$ $\quad card(X) = 3$ $\quad card(\mathcal{P}(X)) = 8$

$\mathcal{P}(X) = \{\emptyset, \{a\}, \{b\}, \{\{3\}\}, \{a,b\}, \{a,\{3\}\}, \{b,\{3\}\}, \{a,b,\{3\}\}\}$

⚠ $card(\mathcal{P}(S)) = 2^{card(S)}$

↳ for each element of S we have one choice to make: - include it in subset - or not

see combinatorics

CARTESIAN PRODUCTS

$A \times B$ is called the Cartesian product of the set A and B and it consists of all the ordered pairs formed from one element of A and another element of B

e.g.: $A = \{\cdot, \cdot\}$ $\quad B = \{\smile, \frown\}$

$A \times B = \{(\cdot, \smile), (\cdot, \frown), (\cdot, \frown), (\cdot, \smile)\}$

"pairs", "tuple"

Parentheses = order matters

$|A| = 2 \quad |B| = 2 \quad |A \times B| = 4$

Jeary saved the day!

Let X, Y be sets with cardinality card(X) and card(Y), what is

$card(X \times Y) = card(X) \times card(Y)$

e.g. $A = \{1, 2\}$ $\quad B = \emptyset$

$A \times B = \{(1,\cdot), (2,\cdot)\}$

$A \times B = \emptyset$

$= \{(1,\cdot), (2,\cdot)\}$

COUNTER-EXAMPLE showing Cartesian product is NOT commutative, $A = \{1\}$ $\quad B = \{2\}$

$A \times B = \{(1,2)\}$ $\quad A \times B \neq B \times A$

$B \times A = \{(2,1)\}$ $\quad$ so not commutative

⚠

$\{\}$ $\neq$ $()$

- BRACES - SETS - order does not matter - repetitions not matter
- parentheses - pairs / tuplet / tuples - order MATTERS - repetitions MATTER

GENERALIZED CARTESIAN PRODUCT

Cartesian products can have any number of components

❓ $X = Bin \times (Bin \times Bin)$, $X = \{(0,(0,0)), (0,(0,1)), (0,(1,0)), \ldots\}$

e.g. $Bin = \{0, 1\}$

$Bin \times Bin \times Bin \times Bin = \{(0,0,0,0), (0,0,0,1), (0,0,1,0), (0,0,1,1), (0,1,0,0), (0,1,0,1), (0,1,1,0), (0,1,1,1), (1,0,0,0), (1,0,0,1), (1,0,1,0), (1,0,1,1), (1,1,0,0), (1,1,0,1), (1,1,1,0), (1,1,1,1)\}$

$card(Bin \times Bin \times Bin \times Bin) = card(Bin) \times card(Bin) \times card(Bin) \times card(Bin) = 2 \times 2 \times 2 \times 2 = 16$

Cartesian product with 4 components

e.g. $A = \{1, 1\}$ $\quad B = \{2\}$ $\quad A \times B = \{(1,2)\}$

For convenience, we usually ignore nesting in Cartesian products. $(a, (b, c)) = (a, b, c)$

PARTITIONNING A SET

When you have a set A that can be partitioned (= "split up" or "divided" or "broken up") into subsets A_i such that

(i) the union of all A_i is A

$A = \bigcup_{i=1}^{3} A_i$

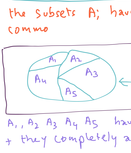
(ii) the A_i's are disjoint = they share no overlap

"for all" possible ways of picking i and j

$\forall i, j \quad if\ i \neq j \Rightarrow A_i \cap A_j = \emptyset$

the subsets A_i have no element in common

e.g.



A_1, A_2, A_3, A_4, A_5 have NO OVERLAP + they completely add up to A

Ex's example

$A = \{1, 1\}$ $\quad B = \{2, 1\}$

$A \times B = \{(1,1), (1,2)\}$

$P(A) = \{\emptyset, \{1\}\}$ $\quad P(B) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$