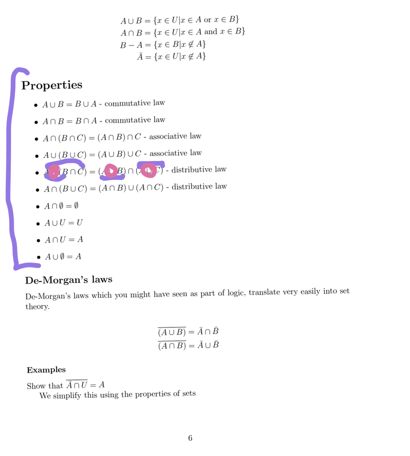
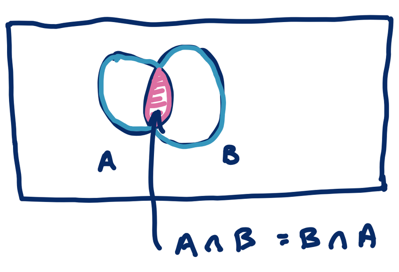
Lecture 3: Set operations continued



Commutative: you can flip operands around the operator

$A \cup B = B \cup A$

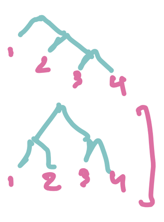
$A \cap B = B \cap A$



$A \cap B = B \cap A$

Associative: you can change the order of parentheses when dealing with a sequence of the same operation

e.g. addition is associative: $1 + 2 + 3 + 4$
$= 1 + (2 + (3 + 4))$
$= (1 + 2) + (3 + 4)$



same operator

$A \cup B \cup C = A \cup (B \cup C)$
$= (A \cup B) \cup C$

$A \cap B \cap C = A \cap (B \cap C)$
$= (A \cap B) \cap C$

Distributive: (the only property where we mix operators)

you can factor or distribute over this operator

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

"intersection is distributive over union"

⚠ Ex. of non-associative operator

SUBTRACTION

$1 - 5 - 5 \neq 1 - (5 - 5)$
$\downarrow \qquad\qquad \downarrow$
$-9 \qquad\qquad 1$

(DIVISION)

$A \cap \emptyset = \emptyset$

$A \cup U = U$

$A \cap U = A$

$A \cup \emptyset = A$

} neutral element for their respective operation

De Morgan's Law

$\overline{A \cup B} = \bar{A} \cap \bar{B}$

flips

$\overline{A \cap B} = \bar{A} \cup \bar{B}$

flips

Example from lecture notes

$\overline{\bar{A} \cap U} = \ldots$

$= A$

More definitions on sets

- **CARDINALITY**: the size (= number of (unique) elements) of a set S and notated $|S|$ or $card(S)$

e.g.

$|\{1, 2, 5\}| = 3$ $\qquad |\mathbb{N}| = \infty$

$card(\{1, 5, 4\}) = 3$

⚠ Cardinality is insensitive to repetitions

$|\{a\}| = |\{a, a, a, a, a, a, a\}| = \underline{1}$

$\boxed{card(\{1, 3, 3, 4\}) = 3}$

- **Powerset**: given a set S, the powerset $P(S)$ is the set of all subsets of S

e.g.: $X = \{5, 9\}$ (a set containing only one element is called a singleton)

$P(X) = \{\emptyset, \{5\}, \{9\}, \underbrace{\{5, 9\}}_{X}\}$

each subset in a powerset is about asking YES/NO whether to include each element

⚠ • $\emptyset$ is always part of the powerset
• S ________________

⚠ $card(P(S)) = 2^{card(S)}$

↳ for each element of S we have one choice to make:
– include it in subset
– or not

more to discuss in combinatorics

CARTESIAN PRODUCTS (& generalized products)

$\{1, 2, 3\}$

$\{\emptyset, \{1\}, \{2\}, \{3\}$
$\{1, 2\}, \{2, 3\},$
$\{1, 3\}, \{1, 2, 3\}\}$

$A \times B$ is called the Cartesian product of the set A and the set B

consists of all the ordered pairs formed from one element of A and one element of B. → ORDER IS IMPORTANT

e.g.: $A = \{ \cdot\cdot, \ddot{\smile} \}$ $\qquad B = \{ \smile, \frown \}$

one element from A
one element from B

$A \times B = \{ \overset{1}{(\cdot\cdot, \smile)}, \overset{2}{(\cdot\cdot, \frown)}, \overset{3}{(\ddot{\smile}, \smile)}, \overset{4}{(\ddot{\smile}, \frown)} \}$

an ordered pair

Cartesian pairs are represented w/ parentheses

$|A| = 2$
$|B| = 2$ $\qquad |A \times B| = 4$

What is the size/cardinality $|A \times B|$ given the size $|A|$ and $|B|$?

$|A \times B| = |A| \times |B|$

⚠ $\{\} \neq ()$
braces — SETS
parentheses — PAIRS, PRODUCTS

GENERALIZED PRODUCT

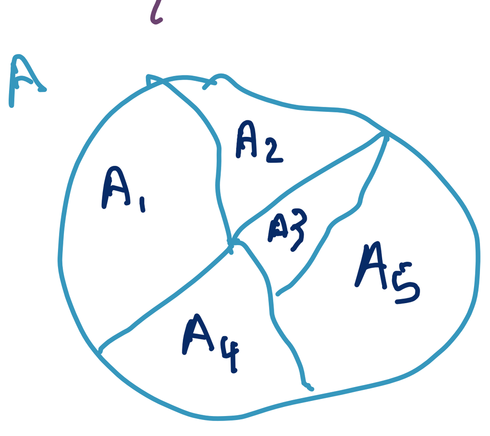
Cartesian products can have any number of components

$Bin = \{0, 1\}$

$\underbrace{Bin \times Bin \times Bin \times Bin}_{\text{Cartesian product w/ 4 components}} = \{(0,0,0,0), (0,0,0,1), (0,0,1,0) \ldots\}$

PARTITIONNING A SET

When you have a set A that can be partitioned (= split = broken up) into subsets A_i such that



- the union of all A_i's is A

$$A = \bigcup_{i=1} A_i$$

- the A_i's are disjoint, they share no overlap

$$\boxed{\forall i, j, \ i \neq j \Rightarrow A_i \cap A_j = \emptyset}$$